SEQUENCES OF FIXED POINT INDICES OF ITERATIONS IN DIMENSION 2

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Abstract. Let ind(f, 0) be the local fixed point index at 0. We show that every sequence of integers which satisfies Dold relations can be realized as \(\{\text{ind}(f^n, 0)\}_{n=1}^{\infty}\), where \(f\) is a continuous self-map of a 2-dimensional disk \(D^2\).

1. Introduction. Let \(f\) be a continuous self-map of a compact ANR \(X\). There are some restrictions on a sequence of fixed point indices \(\{\text{ind}(f^n, X)\}_{n=1}^{\infty}\): it must satisfy congruences established by A. Dold (cf. [6]), called Dold relations. Additional assumptions on \(f\) or \(X\) may give stronger bounds on the shape of \(\{\text{ind}(f^n, X)\}_{n=1}^{\infty}\). For example, if \(f : \mathbb{R}^k \to \mathbb{R}^k\) is a \(C^1\)-map and \(x_0\) is a fixed point of \(f\), then the sequence \(\{\text{ind}(f^n, x_0)\}_{n=1}^{\infty}\) is periodic (cf. [8], [5]). Babenko and Bogatyi proved that every sequence of integers that satisfies Dold relations can be obtained as \(\{\text{ind}(f^n, 0)\}_{n=1}^{\infty}\), where \(f : D^3 \to D^3\) is a homeomorphism and \(D^3\) is a unit disk in \(\mathbb{R}^3\) (cf. [1]). The realization by a homeomorphism is impossible if we replace \(D^3\) by a 2-dimensional unit disk \(D^2\), which is a consequence of the fact that in this case \(\{\text{ind}(f^n, 0)\}_{n=1}^{\infty}\) takes no more than three values (cf. [2], [7], [3]). The question whether the realization is possible for a continuous self-map of \(D^2\) was asked by Babenko and Bogatyi in [1]. This note gives the positive answer to that problem by extending the construction from [1], which was performed for some particular classes of sequences.

2. Dold relations and Dold coefficients. Let \(f : U \to \mathbb{R}^2\), where \(U\) is an open subset of \(\mathbb{R}^2\), be a continuous map such that for each integer \(n > 0\),

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the center 0 is an isolated fixed point for \( f^n \). In this case the fixed point index \( \text{ind}(f^n, 0) \) is well defined for \( f^n \) restricted to a small neighborhood of 0.

Let us recall the definition of the Möbius function:

**Definition 2.1.** The Möbius function \( \mu : \mathbb{N} \rightarrow \mathbb{Z} \) is defined by these three properties:

\[
\mu(1) = 1, \\
\mu(k) = (-1)^r \text{ if } k \text{ is a product of } r \text{ different primes}, \\
\mu(k) = 0 \text{ otherwise}.
\]

**Definition 2.2.** For each natural \( n \) we define an integer \( i_n(f, 0) \) by the equality:

\[
i_n(f, 0) = \sum_{k \mid n} \mu(n/k)\text{ind}(f^k, 0).
\]

The following congruences were found by A. Dold (cf. [6]):

**Theorem 2.3 (Dold relations).** For every \( n \in \mathbb{N} \), there is:

\[
i_n(f) \equiv 0 \pmod{n}.
\]

We will call the numbers \( A_n(f, 0) = \frac{1}{n}i_n(f, 0) \) Dold coefficients. By the Möbius inversion formula (cf. [4]) we obtain:

\[
\text{ind}(f^n, 0) = \sum_{k \mid n} kA_k(f, 0).
\]

Let us notice that another way of expressing Dold relations is to say that each \( A_n(f, 0) \) is an integer. As we have a one-to-one correspondence between elements of \( \{\text{ind}(f^n, 0)\}_{n=1}^\infty \) and \( \{A_n(f, 0)\}_{n=1}^\infty \) we may reformulate the basic question of realization in the terms of Dold coefficients.

Instead of asking whether for a given sequence of integers \( \{c_n\}_{n=1}^\infty \) with \( n \sum_{k \mid n} \mu(n/k)c_k \) (Dold relation must be satisfied) there exists a map \( f \) such that \( c_n = \text{ind}(f^n, 0) \) for each \( n \), we may ask whether for a given arbitrary sequence of integers \( \{b_n\}_{n=1}^\infty \) there is a map \( f \) such that \( b_n = A_n(f, 0) \) for each \( n \).

**Lemma 2.4.** (cf. [1]) Let \( f : X \rightarrow X, \ g : Y \rightarrow Y \) be continuous maps of absolute neighbourhood retracts \( X, \ Y \) with isolated fixed points \( p \) and \( q \), respectively. Then for the fixed point \( (p, q) \) of the map: \( f \vee g : X \vee Y \rightarrow X \vee Y \), where \( X \vee Y \) is a fan (bouquet) of spaces \( X \) and \( Y \), there is:

\[
\text{ind}(f \vee g, (p, q)) = \text{ind}(f, p) + \text{ind}(g, q) - 1, \\
A_1(f \vee g, (p, q)) = A_1(f, p) + A_1(g, q) - 1, \\
A_n(f \vee g, (p, q)) = A_n(f, p) + A_n(g, q),
\]

for \( n > 1 \).
3. The realization of an arbitrary sequence of integers. Let $A_d$ be an arbitrary integer, $d > 1$. Consider two cases: (1) $A_d > 0$. Let $T_d$ be a fan (see an example of $T_3$ in Picture 1) composed of $d$ isometric isosceles triangles with one common vertex at 0. Define a map $f_d : T_d \to T_d$ as the composition $f_d = g_d \theta_d$, where $\theta_d$ is a preserving orientation isometry which maps each triangle in counter-clockwise order on the nearest one ($\theta_d^d = id$), $g_d$ is equal to $g$ on each triangle of the fan, where $g$ is time-one map of the flow given in Picture 3a ($\text{ind}(g, 0) = 2$).

By Lemma 2.4 and the definition of fixed point index, we obtain:

$$\text{ind}(f^n_d, 0) = \begin{cases} 1 & \text{if } d \nmid n, \\ 1 + d & \text{if } d | n. \end{cases}$$

It easy to check, using Definition 2.2 that in terms of Dold coefficients this is equivalent to:

$$A_n(f_d, 0) = \begin{cases} 1 & \text{if } n = 1 \text{ or } n = d, \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 3.1.** We may look at the above construction as at a kind of inductive procedure. We have 0, the fixed point itself; it has index equal to 1. Then, every time we add a single triangle from the family $T_d$ with $\text{ind}(g, 0) = 2$, we must, by Lemma 2.4, add the number $(2-1)$ to index of $f^n_d$, where $d | n$.

Now we take a fan $T_{d,A_d} = \bigvee_{d=1}^{A_d} T_d$ which consists of $A_d$ copies of $T_d$, with $f_{d,A_d} = \bigvee_{d=1}^{A_d} f_d : T_{d,A_d} \to T_{d,A_d}$. Then, again by Lemma 2.4, there is:

$$A_n(f_{d,A_d}, 0) = \begin{cases} 1 & \text{if } n = 1, \\ A_d & \text{if } n = d, \\ 0 & \text{otherwise.} \end{cases}$$

(2) $A_d \leq 0$. We repeat the construction from [1], in which segments are used instead of triangles. Let $I_d$ be a fan composed of $d$ segments of the same length with 0 as their only common point (see an example of $I_3$ in Picture 2). Define $f_d : I_d \to I_d$ as the composition $f_d = h_d \theta_d$, where $\theta_d$ isometrically maps each segment on the nearest one in the counter-clockwise order, $h_d$ is equal to $h$ on each segment, where $h$ is time-one map of the flow given in Picture 4a ($\text{ind}(h, 0) = 0$). Then, by Lemma 2.4

$$\text{ind}(f^n_d, 0) = \begin{cases} 1 & \text{if } d \nmid n, \\ 1 - d & \text{if } d | n. \end{cases}$$
Now we take a fan $I_{d,A_d} = \bigvee_{d=1}^{\lvert A_d \rvert} I_d$ which consists of $\lvert A_d \rvert$ copies of $I_d$ ($I_{d,0} = \{0\}$), with $f_{d,A_d} = \bigvee_{d=1}^{\lvert A_d \rvert} f_d : I_{d,A_d} \to I_{d,A_d}$. Then, again by Lemma 2.4, there is:

$$A_n(f_{d,A_d}, 0) = \begin{cases} 1 & \text{if } n = 1, \\ -1 & \text{if } n = d, \\ 0 & \text{otherwise}. \end{cases}$$

For $d = 1$, we introduce a slight change: if $A_1 > 0$, then we define $T_{1,A_1}$ as a fan composed of $A_1 - 1$ triangles, if $A_1 \leq 0$, then we define $I_{1,A_1}$ as composed of $\lvert A_1 \rvert + 1$ segments, in both cases taking $f_{1,A_d}$ in the same way as for $d \neq 1$.

**Remark 3.2.** Notice that if we replace the map $g$ and $h$ by the time-one map of the flow in Pictures 3b and 4b, respectively, then all Dold coefficients
for \( n > 1 \) will equal zero, \( A_n(f_d, 0) = 0 \). It is a consequence of the fact that an addition of another triangle or segment with \( \text{ind}(g, 0) = 1 \) or \( \text{ind}(h, 0) = 1 \) does not change the index of \( f_d^n \). By Lemma 2.4 it will be constant and equal to 1, so the inductive procedure of enlarging the index, described in Remark 3.1, does not work.

Now assume that an arbitrary sequence of integers \( \{A_d\}_{d=1}^\infty \) is given. For each \( A_d \) we choose an appropriate \( X_{d,A_d} \), where \( X_{d,A_d} \in \{T_{d,A_d}, I_{d,A_d}\} \) in such a way that it is placed in the planar sector defined in polar coordinates \( \phi, \rho \) by: \( |\phi - \frac{\pi}{2d}| < \frac{x}{2d}, \rho < \frac{1}{4} \) and the height from the corner 0 of each \( T_d \) or the length of each \( I_d \) is equal to \( \frac{1}{2d} \).

Let us take \( f_\infty = \bigvee_{d=1}^\infty f_d, A_d \), which is a self-map of \( K_\infty = \bigvee_{d=1}^\infty X_{d,A_d} \).

Let us state the following well-known fact:

**Lemma 3.3.** Let \( X \) be a compact AR, \( F : X \to X \), continuous. If \( p \) is the only fixed point of \( F \), then \( \text{ind}(F, p) = 1 \).

For an arbitrary given \( n \) we represent \( f_\infty \) as \( F_n \vee G_{n+1} \), where \( F_n = \bigvee_{d=1}^n f_d, A_d \) is a self-map of \( \bigvee_{d=1}^n X_{d,A_d} \), \( G_{n+1} = \bigvee_{d=n+1}^\infty f_d, A_d \) is a self-map of \( \bigvee_{d=n+1}^\infty X_{d,A_d} \). Notice that \( \text{Fix}(G_{n+1}) = \{0\} \) for each \( k \leq n \), so by Lemma 3.3 we get that \( \text{ind}(G_{n+1}, 0) = 1 \) for \( k \leq n \). As a consequence we obtain that \( A_1(G_{n+1}, 0) = 1, A_k(G_{n+1}, 0) = 0 \) for each \( 1 < k \leq n \). By Lemma 2.4 we have: \( A_k(f_\infty, 0) = A_k(F_n, 0) + A_k(G_{n+1}, 0) \) for \( k > 1 \) and \( A_1(f_\infty, 0) = A_1(F_n, 0) + A_1(G_{n+1}, 0) - 1 \), thus \( A_k(f_\infty, 0) = A_k(F_n, 0) \) for \( k \leq n \).

Applying again Lemma 2.4 we get: \( A_k(F_n, 0) = A_k \) for \( k \leq n \).

Finally we define \( f : D^2 \to D^2 \) by \( f = f_\infty \circ r \), where \( r : D^2 \to K_\infty \) is retraction and get that \( A_k(f, 0) = A_k \), for \( k = 1, 2, 3, \ldots \). This completes the construction.

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**References**


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