EXPLICIT FORMULAS FOR THE ROOTS OF THE EQUATION

\[ e^{ix} = (i - \frac{x}{2})/(i + \frac{x}{2}) \]

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Abstract. We derive explicit formulas for all roots of the equation in title.

The equation above (here and thereafter \(i\) is the imaginary unit) was studied by Konstanty Holly. He proved in [3] that its real roots coincide with those of another transcendental equation. Our goal here is to solve the equation in a closed form. Namely, we shall prove that it has real roots only and derive an explicit integral representation for every root of the equation. Such formulas have been proven to be useful for studying the asymptotic behavior of the roots when parameters tend to their extremal values, and also for numerical calculation of the roots. We use the approach of [1, 2, 4] and consider a slightly more general equation

\[ e^{iaz} = \frac{i - z}{i + z}, 0 < a < 4. \]

When the parameter \(a = 2\), equation (1) reduces to the original equation in the title after substitution \(z = x/2\).

To start with, by taking the logarithms of both sides of the equation, we reduce (1) to an infinite set of the equations

\[ z = -\frac{i}{a} \ln \frac{i - z}{i + z} + \frac{2\pi}{a}, k = 0, \pm 1, \pm 2, \ldots \]

These equations look more complicated than (1), however, the method employed is based on the observation that for any integer \(k\), equation (2) has an at most finite number of roots. We shall show that for each \(k \neq 0\) the equation

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(2) has exactly one root, and only for \( k = 0 \) (2) has three roots. To do that, we consider analytic functions

\[
f_k(z) = z + \frac{i}{a} \ln \frac{i - z}{i + z} - \frac{2\pi}{a} k, \quad k = 0, \pm 1, \pm 2, \ldots
\]

Straightforward graph sketching shows that (1) has no roots on rays \( z = iy, \; 1 < y < \infty \) and \(-\infty < y < -1\). Therefore, to select single-valued branches of the functions \( f_k \), we cut the complex plane along these rays. Denote the slit domain by \( D \) and fix a single-valued branch of the logarithmic function in \( D \) under the condition \( \ln 1 = 0 \). Now, every \( f_k(z) \), \( k = 0, \pm 1, \pm 2, \ldots \), is a single-valued holomorphic function in \( D \) and the equation (1) is equivalent to the alternative of the equations \( f_k(z) = 0 \), \( k = 0, \pm 1, \pm 2, \ldots \).

To find the number of roots of the equation \( f_k(z) = 0 \), we apply the argument principle. First, we have to calculate the boundary values of \( f_k \) at the shoreline of the cuts. Consider the shoreline \( z = iy + 0, \; y > 1 \). Let \( \gamma \subset D \) be a smooth path connecting the origin with a point \( z \) and, as usual, denote by \( \Delta_\gamma \arg \zeta \) the increment of the argument of a point \( \zeta \) when the point runs from 0 to \( z \) along \( \gamma \). Since \( \ln 1 = 0 \), we have \( \Delta_\gamma \arg(i - z) = \pi, \Delta_\gamma \arg(i + z) = 0 \),

\[
\ln \frac{i - iy}{i + iy} = \ln \frac{y - 1}{y + 1} + i\{\Delta_\gamma \arg(i - z) - \Delta_\gamma \arg(i + z)\} = \ln \frac{y - 1}{y + 1} + \pi i,
\]

and finally, for \( y > 1 \)

\[
f_k(iy + 0) = iy + \frac{i}{a} \ln \frac{y - 1}{y + 1} - \frac{\pi}{a} (2k + 1).
\]

In the same way,

\[
f_k(iy - 0) = iy + \frac{i}{a} \ln \frac{y - 1}{y + 1} - \frac{\pi}{a} (2k - 1), \text{ for } y > 1,
\]

\[
f_k(iy + 0) = iy + \frac{i}{a} \ln \frac{y - 1}{y + 1} - \frac{\pi}{a} (2k + 1), \text{ for } y < -1,
\]

\[
f_k(iy - 0) = iy + \frac{i}{a} \ln \frac{y - 1}{y + 1} - \frac{\pi}{a} (2k - 1), \text{ for } y < -1.
\]

Next, consider a closed contour \( \omega \) in \( D \) consisting of two arcs of radius \( R \) in the right and left half-planes respectively, of two small arcs of radius \( \varepsilon \) centered at the points \( \pm i \), and of four vertical segments joining them along the cuts (Figure 1).
First, we study the roots of $f_0(z)$. When a point $z$ runs through the contour $\omega$ and $\kappa = 0$, the corresponding point $f_0(z)$ circumscribes the closed contour $\Omega_0$ (Figure 2) where two dashed vertical lines have the equations $Re\ z = \pm \pi/2$. 
From Figure 2 we see that the contour $\Omega_0$ circumscribes the origin three times, that is, $\Delta_w \arg f_0(z) = 6\pi$. According to the argument principle, $f_0$ has three roots in $D$; obviously, one of them is $z = 0$. Next, $f_0(-z) = -f_0(z)$, thus, we denote the two other roots by $\xi_0$ and $-\xi_0$.

If $k \neq 0$, the function $f_k$ differs from $f_0$ by a real addend $-\frac{2\pi}{a} k$. Therefore, when $z$ runs through $\omega$, the point $f_k(z)$ runs through the contour $\Omega_k = \Omega_0 - \frac{2\pi}{a} k$, that is, $\Omega_k$ is the contour $\Omega_0$ shifted by $-\frac{2\pi}{a} k$ in the horizontal direction. Since $| -\frac{2\pi}{a} \frac{k}{a} | > \frac{\pi}{2}$ when $k \neq 0$ and $0 < a < 4$, any contour $\Omega_k$, $k \neq 0$, circumscribes the origin only once. Thus, each function $f_k$, $k \neq 0$, has one root only; we denote it by $\xi_k$.

To obtain explicit formulas for the roots $\xi_k$, $k \neq 0$, we consider an integral

$$I_k = \oint_{\omega} \frac{\zeta - \xi_k}{\zeta^2 f_k(\zeta)} \, d\zeta$$

along the same closed contour $\omega$. The integrand has a removable singularity at $\zeta = \xi_k$ and a double pole at $\zeta = 0$. Calculating the residue at the pole, we have

$$I_k = \frac{ia(a-2)}{2\pi k^2} \left( \xi_k - \frac{a}{k} i \right).$$  \hspace{1cm} (7)

On the other hand, we deform the contour $\omega$ until it coincides with the boundary of the slit plane $D$. The integrals along the large arcs of the radius $R$ vanish as $R \to \infty$ by virtue of Watson's lemma. The integrals along the small arcs of radius $\varepsilon$ vanish as $\varepsilon \to 0$ due to a direct estimation. Using the formulas (3)–(6), we get the equation
\[ I_k = \int_1^\infty \frac{(iy - \xi_k)idy}{y^2[iy + \frac{i}{a} \ln \frac{y-1}{y+1} - \frac{\pi}{a} (2k + 1)]} - \]
\[ \int_1^\infty \frac{(iy - \xi_k)idy}{y^2[iy + \frac{i}{a} \ln \frac{y-1}{y+1} - \frac{\pi}{a} (2k - 1)]} + \]
\[ \int_{-\infty}^1 \frac{(iy - \xi_k)idy}{y^2[iy + \frac{i}{a} \ln \frac{y-1}{y+1} - \frac{\pi}{a} (2k - 1)]} - \]
\[ \int_{-\infty}^1 \frac{(iy - \xi_k)idy}{y^2[iy + \frac{i}{a} \ln \frac{y-1}{y+1} - \frac{\pi}{a} (2k + 1)]} = \]
\[ \frac{2\pi i}{a} \xi_k \left[(2k + 1) \int_1^\infty \frac{dy}{y^2[(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 + \frac{\pi^2}{a^2} (2k + 1)^2]} \right] - \]
\[ (2k - 1) \int_1^\infty \frac{dy}{y^2[(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 + \frac{\pi^2}{a^2} (2k - 1)^2]} \right] + \]
\[ 2i \left\{ \int_1^\infty \frac{(y + \frac{1}{a} \ln \frac{y-1}{y+1})dy}{y[(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 + \frac{\pi^2}{a^2} (2k + 1)^2]} - \right\}
\[ \int_1^\infty \frac{(y + \frac{1}{a} \ln \frac{y-1}{y+1})dy}{y[(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 + \frac{\pi^2}{a^2} (2k - 1)^2]} \right\} = \]
\[ \frac{4\pi i}{a} \xi_k A_k - \frac{16\pi^2 ik}{a^2} B_k, \]

where

\[ A_k = \int_1^\infty \frac{[(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 - \frac{\pi^2}{a^2} (4k^2 - 1)]dy}{y^2[(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 + \frac{\pi^2}{a^2} (2k + 1)^2][(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 + \frac{\pi^2}{a^2} (2k - 1)^2]} \]

and

\[ B_k = \int_1^\infty \frac{(y + \frac{1}{a} \ln \frac{y-1}{y+1})dy}{y[(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 + \frac{\pi^2}{a^2} (2k + 1)^2][(y + \frac{1}{a} \ln \frac{y-1}{y+1})^2 + \frac{\pi^2}{a^2} (2k - 1)^2]} \].

From (7) and (8), we obtain the desired expression for the roots \( \xi_k, k \neq 0 \):

\[ \xi_k = \frac{2\pi k(16\pi^2 k^2 B_k - a^3)}{a(8\pi^2 k^2 A_k - a^2(a - 2))}. \]

It is clear from (9) that all these roots are real, moreover, \( \xi_{-k} = -\xi_k \).

To this end, we are to calculate the roots \( \pm \xi_0 \). Direct differentiation shows that, when \( a = 2 \), \( f_0(z) \) has a triple root at \( z = 0 \); that is, \( \pm \xi_0 = 0 \) if \( a = 2 \). If
. \, a \neq 2, \text{ we consider an integral }

\[ I_0 = \int_\omega \frac{\zeta^2 - \xi_0^2}{\zeta^4 f_0(\zeta)} \, d\zeta, \]

which has a five-fold root at \( \zeta = 0 \). After similar calculations, we arrive at the equation

\[ I_0 = -\frac{4\pi ai}{3(a - 2)^2} - \frac{4\pi a(9a - 8)i}{45(a - 2)^3} \xi_0^2 = -\frac{4\pi i}{a} I_{00}, \]

where

\[ I_{00} = \int_1^\infty \frac{(y^2 + \xi_0^2)dy}{y^4[(y + \frac{1}{a} \ln \frac{y - 1}{y + 1})^2 + \frac{\pi^2}{a^2}]} . \]

From the last two equations,

\[ \xi_0^2 = 15(a - 2) \frac{a^2 - 3A_0(a - 2)^2}{45B_0(a - 2)^3 - a^2(9a - 8)}, \]

where

\[ A_0 = \int_1^\infty \frac{dy}{y^2[\left(y + \frac{1}{a} \ln \frac{y - 1}{y + 1}\right)^2 + \frac{\pi^2}{a^2}]} \]

and

\[ B_0 = \int_1^\infty \frac{dy}{y^4[\left(y + \frac{1}{a} \ln \frac{y - 1}{y + 1}\right)^2 + \frac{\pi^2}{a^2}]} . \]

Therefore, if \( a \neq 2 \), then \( f_0(z) \) has, in addition to \( z = 0 \), either two symmetrical real roots or two conjugate pure imaginary roots, depending on the sign of the last expression for \( \xi_0^2 \).

In particular, from this analysis and from the equation (9) with \( a = 2 \) we conclude that the original equation in the title has the triple root at \( x = 0 \) and the infinite sequence of real symmetrical roots (we use the relation \( x_k = 2\xi_k \))

\[ x_k = \frac{2(2\pi^2 k^2 B_k - 1)}{\pi k A_k}, \quad k = \pm 1, \pm 2, \ldots , \]

where \( A_k \) and \( B_k \) were defined above.

It should finally be noted that the integrals \( A_k \) and \( B_k \) converge slowly. However, the substitution \( y = (e^t + 1)/(e^t - 1) \) reduces them to exponentially converging integrals. Thus, after this substitution, it took about 40 seconds to calculate \( \xi_0 \) with eight decimal digits by making use of the MAPLE V on a slow PC with a = 166 MHz processor. However, the calculations using the original expression (9) failed and were interrupted after 10 minutes.
References


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