ANALYTIC SOLVABILITY OF SOME SINGULAR DIFFERENTIAL OPERATORS

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Abstract. The subject of this note is the solvability of the differential operator $Lu = D_X u - Bu$ where $X$ is a vector field. This is a continuation of [2], where the $C^\infty$ solutions in the singular case were discussed. We give conditions on the existence of formal and analytic solutions at a critical point of $X$. The results generalize Siegel's theorem in its part concerning homological equation. A lemma giving an estimate for the norms of bounded linear operators in Hilbert spaces is proved. There are also geometrical consequences, since the Lie derivative operator is of this type.

1. Formal solvability.

Notation. We denote by $C^k(n, m)$, for $k = 0, 1, 2, \ldots, \infty, \omega$, the space of germs at the origin $0 \in \mathbb{R}^n$ of $C^k$ maps (analytic if $k = \omega$) from $\mathbb{R}^n$ to $\mathbb{R}^m$. We consider a linear differential operator $L : C^{k+1}(n, m) \to C^k(n, m)$ defined by $Lu = D_X u - Bu$, where $X \in C^1(n, n)$ is a vector function, $B \in C^k(n, m \times m)$ is a matrix function, and $D_X u$ stands for the directional derivative of $u$ in direction $X$. The singularity assumption is $X(0) = 0$.

The main theorem of this part is the following

Theorem 1. The partial differential equation

\begin{equation}
D_X u - Bu = f, \quad u(0) = 0, \tag{1.1}
\end{equation}

has a formal solution $u$ at $x = 0$ for each $f \in C^\infty(n, m), f(0) = 0$, if and only if no linear combination, with non-negative integer coefficients, of the
eigenvalues of the Jacobian matrix $DX(0)$ is an eigenvalue of $B(0)$ (in other words: $B(0)$ is not in resonance with $DX(0)$). The solution is unique.

PROOF. Denote by $X_i^i, f^p, u^p_\mu$, $1 \leq i \leq n, 1 \leq p \leq m, |\mu| \geq 1$, the coefficients in the Taylor formal expansions of $X, f, u$ respectively. In coordinates, equation (1.1) writes

\begin{equation}
\sum_{1 \leq i \leq n} X_i^i(x)D_i u^p - \sum_{1 \leq q \leq m} B_q^p(x)u^q = f^p(x).
\end{equation}

We set $X_i^i(x) = \sum_j a_j^i x^j + o(x), \ a_j^i = D_j X_i^i(0)$; we may assume that the Jacobian matrix $(a_j^i)$ is in Jordan triangular form. Inserting this and the formal expansions of $f$ and $u$ into (1.2) we find

\begin{equation}
\sum_i (\sum_j a_j^i x^j + o(x)) \sum_{|\beta| \geq 1} \beta_i u_\beta^p x^{\beta - \lambda_i} - \sum_q (B_q^p(0) + o(1)) \sum_{|\beta| \geq 1} u_\beta^q x^{\beta} = \sum_{|\gamma| \geq 1} f_\gamma^p x^{\gamma}.
\end{equation}

Hence

\begin{equation}
\sum_{i,j} (a_j^i + o(1)) \sum_{\beta} \beta_i u_\beta^p x^{\beta - \lambda_i + \lambda_j} - \sum_q (B_q^p(0) + o(1)) \sum_{|\beta| \geq 1} u_\beta^q x^{\beta} = \sum_{|\gamma| \geq 1} f_\gamma^p x^{\gamma}.
\end{equation}

Comparing the coefficients at $x^{\gamma}$ in (1.5), with $|\gamma| = k \geq 1$, yields

\begin{equation}
\sum_{1 \leq i,j \leq n}^{1 \leq i,j \leq n} \beta_i a_j^i u_\beta^p - \sum_q B_q^p(0) u_\gamma^q = f_\gamma^p, \quad (p = 1, \ldots, m, |\gamma| = k).
\end{equation}

Substituting $u_\beta^p = \sum_{|\mu| = k} \delta_\mu^p u_\mu^p$ in (1.4), we obtain

\[\sum_{|\mu| = k} u_\mu^p (\sum_{1 \leq i,j \leq n}^{1 \leq i,j \leq n} \beta_i a_j^i \delta_\mu^p) - \sum_{1 \leq q \leq m} B_q^p(0) u_\gamma^q = f_\gamma^p.\]

In the matrix form this reads

\begin{equation}
UM - B(0)U = F,
\end{equation}

where $U = (u_\mu^p)$ denotes the matrix of the unknowns, $M$ is a square matrix with the elements

\[M_\mu^\gamma = \sum_{i,j} (\gamma + 1_i - 1_j) a_j^i \delta_\gamma^{\mu + 1_i - 1_j},\]

and $F = (f_\gamma^p)$, with $|\mu| = |\gamma| = k$. 
Claim \( M \) has eigenvalues \( (\gamma, \lambda) = \sum_{i=1}^{n} \gamma_{i} \lambda_{i} \).

In fact, using the triangular form \( a_{ij} = \lambda_{i} \delta_{j}^{i} + \epsilon_{i} \delta_{j}^{i-1} \) we get

\[
M_{\gamma}^{\mu} = (\gamma, \lambda) \delta^{\mu}_{\gamma} + \sum_{i}(\gamma - 1)_{i+1} + 1_{i}) \epsilon_{i} \delta_{\gamma - 1, i+1} + 1_{i}.
\]

If we order \( \mu, \gamma \) lexicographically we see that the coefficient matrix \( M \) is triangular with \( \binom{n+k-1}{k} \) eigenvalues \( (\gamma, \lambda) \).

The matrix equation (1.5) is a Cramer linear system if and only if the matrices \( M \) and \( B(0) \) have no common eigenvalues (cf. [1]). This proves the theorem. \( \square \)

Remark. Let us mention that under a weaker condition: \( (\gamma, \lambda) \) is not an eigenvalue of \( B(0) \) for \( |\gamma| \leq k < \infty \), one obtains a version of Theorem 1 for \( f \in C^{k}(n, m) \) and formal solutions of finite order \( k \).

2. Analytic solvability. Now we assume that \( X;B \) and \( f \) are real analytic maps in a neighborhood of the origin \( 0 \in \mathbb{R}^{n} \). With notation of the previous section we define a stronger "no resonance" condition as follows

Definition. We say that a system of complex numbers \( (\lambda_{1}, ..., \lambda_{n}) \in \mathbb{C}^{n} \) is of type \( (C, \nu, B) \), where \( C > 0, \nu > 0 \) are constants and \( B \subset \mathbb{C} \), if

\[
|\left(\alpha, \lambda\right) - b| \geq C|\alpha|^{-\nu}
\]

for every multiindex \( \alpha \in \mathbb{N}^{n} \) and each \( b \in B \).

Theorem 2. If the spectrum \( \lambda = (\lambda_{1}, ..., \lambda_{n}) \) of \( DX(0) \) is of type \( (C, \nu, B) \) with \( B = \text{Spect} \, B(0) \), the equation (1.1) has a (unique) analytic solution near the origin.

Proof. From (2.1) it follows that \( B(0) \) is not in resonance with \( DX(0) \). Therefore, by Theorem 1 there exists a formal solution \( u^{p} = \sum_{|\mu| \geq 1} u_{\mu}^{p} x^{\mu}, \)

\( 1 \leq p \leq m. \)

In (1.5), the linear operator \( P : U \rightarrow UM - B(0)U \) has eigenvalues \( (\alpha, \lambda) - b \neq 0. \) So \( P \) is invertible and its inverse \( Q = P^{-1} \) has eigenvalues \( ((\alpha, \lambda) - b)^{-1}, |\alpha| = k, b \in B. \)

Lemma 1. Let \( Q \) be a bounded and logarithmizable matrix in a complex Hilbert space. Then

\[
\|Q\| \leq \sup\{|\lambda|; \lambda \in \text{Spect} \, Q\}
\]

where \( \|Q\| = \sup_{|x| = 1} |Qx| \).

In particular, the estimate (2.2) holds for every finite non-singular matrix.
PROOF. Suppose $Q = e^A$ for a bounded matrix $A$ (a bounded linear operator). By Heinz inequality [3]

$$
\|Q\| \leq e^{\sup_{|x|=1} (A^ox, \bar{x})},
$$

where $A^o = \frac{1}{2}(A + A^*)$. On the other hand (Wintner [5])

$$
\sup_{|x|=1} (A^ox, \bar{x}) = \sup\{\text{Spect } A^o\} = \sup\{\text{Re}(\text{Spect } A)\}.
$$

Also (Pazy [4])

$$
e^{\text{Spect } A} \subset \text{Spect } e^A.
$$

For $\lambda = e^{\alpha+i\beta}$ one has $e^{\text{Re}(\alpha+i\beta)} = |\lambda|$. Therefore, from (2.3)-(2.5) we obtain

$$
\|Q\| \leq e^{\sup\{\text{Re}(\text{Spect } A)\}} = \sup e^{\text{Re}(\text{Spect } A)} \leq \sup\{|\lambda|; \lambda \in \text{Spect } Q\}
$$

which was to be proved.

Note that any non-singular finite complex matrix is logarithmizable. \qed

Applying the Lemma and estimation (2.1) for $|\alpha| = k$ one gets

$$
\|Q\| \leq C^{-1} k^\nu.
$$

The solution of matrix equation (1.5) can be written as $U = Q F$, so by (2.6)

$$
|U| \leq C^{-1} k^\nu |F|
$$

Since $f$ is analytic at $x = 0$, the Cauchy inequalities yield

$$
|F| = \max_{1 \leq \gamma \leq m} |f^{(\gamma)}_{\gamma}| \leq C_1 \rho^{-k},
$$

where $\rho$ is less than the radius of convergence of the Taylor expansion of $f$.

Using Stirling's formula we estimate the number \((n+k-1)\) of homogeneous terms of order $k$ in $\sum_{\mu} u_{\mu} x^{\mu}$:

$$
\binom{n+k-1}{k} \leq \frac{(n+k-1)^k}{k!} = \frac{(n+k-1)}{k^k e^{-k} \sqrt{2\pi k e^{1/k} \frac{g}{12k}}}
$$

$$
\leq \left(\frac{n+k-1}{k}\right)^k e^k = \left[\left(1 + \frac{n-1}{k}\right)^\frac{k-1}{n-1}\right]^{n-1} e^k
$$
\[ \leq 3^{n+k-1}. \]

Applying the inequalities (2.6)-(2.9) we find that

\[ \left| \sum_{|\mu|=k} u_{\mu}^p x^\mu \right| \leq 3^{n+k-1} |U| |x|^k \leq 3^{n+k-1} C^{-1} k^\nu C_1 \rho^{-k} |x|^k \leq C_2 \left| \frac{4x}{\rho} \right|^k. \]

It follows that the solution series

\[ \sum_{|\mu|\geq1} u_{\mu}^p x^\mu, \quad p = 1, \ldots, m, \]

are convergent in the ball \(|x| < \frac{\rho}{4}\). This completes the proof of the theorem.

\[ \square \]

References


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