GENERAL CONDITIONS FOR EXISTENCE
OF A SOLUTION
OF A SEMILINEAR PROBLEM

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Abstract. This paper is devoted to the investigation of the abstract semilinear
initial value problem
\[
\begin{aligned}
\frac{du}{dt} &= Au + f(t, u) \\
u(0) &= x
\end{aligned}
\]
in a reflexive Banach space. We give conditions for existence of the solution.

1. Introduction. Let $X$ be a real reflexive Banach space and let $A : X \supset D \rightarrow X$ be a linear closed densely defined operator. Let $f : [0, T] \times X \rightarrow X$ be a continuous function.
We consider the abstract initial value problem
\[
\begin{aligned}
\frac{du}{dt} &= Au + f(t, u) \quad t \in [0, T] \\
u(0) &= x
\end{aligned}
\]
(1)
x \in X.

Our purpose is to prove a theorem on existence and uniqueness of the solution
of (1). In [7] an existence and uniqueness theorem was proved under assumption
that $f$ satisfies the Lipschitz condition. Instead of the Lipschitz condition
we only assume that $f$ has bounded variation.

2. Preliminaries.

Definition 1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of $X$. By $X^*$ we denote the Banach space of linear bounded forms on $X$. We say that $\{x_n\}_{n=1}^{\infty}$
. converges weakly if for every \( x^* \in X^* \) the sequence \( \{< x_n, x^* >\}_{n=1}^\infty = \{x^*(x_n)\}_{n=1}^\infty \) is convergent in \( \mathbb{R} \) when \( n \to \infty \). We say that \( \{x_n\}_{n=1}^\infty \) converges weakly to \( x \in X \) if the sequence \( \{< x_n - x, x^* >\}_{n=1}^\infty \) converges to 0, when \( n \to \infty \), for every \( x^* \in X^* \).

**Lemma 1** ([5], Th. 10.5, P. 40). If \( X \) is a reflexive Banach space and \( A : D \to X \) is a closed linear operator densely defined, then the operator \( A^* \) adjoint to \( A \) is densely defined in \( X^* \).

**Lemma 2.** If \( X \) is a reflexive Banach space, then every weakly convergent sequence \( \{x_n\} \subset X \) is weakly convergent to an element of \( X \).

The proof is similar as in the case of Hilbert space ([4], Corollary 3, p. 98).

**Lemma 3** ([2], Ex. 5.12, P. 165). Let \( A : D \to X \) be a closed linear operator in \( X \). If \( \{x_n\} \subset D \) is weakly convergent to \( x \) and \( \{Ax_n\} \) is weakly convergent to \( y \), then \( x \in D \) and \( Ax = y \).

**Lemma 4** ([5], Th. 2.4, Pp. 4–5). Let \( \{S(t)\} \) be a \( C_0 \) semigroup on a Banach space \( X \) and let \( A : D \to X \) be its generator.

Then

\[
(a) \quad \int_r^t S(s)xds \in D \text{ and } A\left(\int_r^t S(s)xds\right) = S(t)x - S(r)x \quad \text{for } x \in X \text{ and } r, t \in [0, \infty)
\]

\[
(b) \quad \frac{d}{dt}(S(t)x) = AS(t)x = S(t)Ax \quad \text{for } x \in D, \ t \in [0, \infty).
\]

**Lemma 5** ([2], Lem. III 1.31). Let \( \{x_n\} \) be a bounded sequence of elements of \( X \). Let \( F^* \subset X^* \) be a dense subset of \( X^* \) such that \( \{< x_n, x^* >\} \) converges for \( x^* \in F^* \). Then \( \{x_n\} \) converges weakly.

**Lemma 6** ([6], Th. 2.1, P. 8). Let \( u : [0, T] \to [0, \infty) \) be a continuous function. If there exist \( \alpha, \beta \geq 0 \) such that

\[
u(t) \leq \alpha + \beta \int_0^t u(s)ds \quad \text{for } t \in [0, T],
\]

then

\[
u(t) \leq \alpha e^{\beta t} \quad \text{for } t \in [0, T].
\]
Lemma 7 ([2], Lem. III 3.7). Let \( K \subset X \) be a compact subset of \( X \). If \( \{T_n\}_{n=1}^{\infty} \) is a sequence of bounded operators on \( X \) strongly convergent to a bounded operator \( T \), then \( \{T_n\}_{n=1}^{\infty} \) converges uniformly on \( K \).

Definition 2. We say that \( f : [a, b] \rightarrow X \) has bounded variation if

\[
W_b^a(f) = \sup \left\{ \sum_{j=1}^{n} \| f(t_j) - f(t_{j-1}) \| : a \leq t_0 < \ldots < t_n \leq b \right\}
\]
is finite.

Lemma 8 ([3], Th. 4, P. 23). Let \( f : [a, b] \rightarrow X \) be a continuous function. If \( f \) has bounded variation, then the function \( [a, b] \ni t \rightarrow W_t^a(f) \in [0, \infty) \) is uniformly continuous.

Lemma 9 ([3], Ex. 10, P. 217). Suppose that \( f : [a, b] \rightarrow X \) is a continuous function and \( \{\{s_j^l\}_{j=0}^{k_l}\}_{l=1}^{\infty} \) is a normal sequence of partitions of \([a, b]\) \((a = s_0^l < s_1^l < \ldots < s_{k_l}^l = b, \text{ for } l \in \{1, 2, 3, \ldots\}\), such that

\[
\lim_{l \rightarrow \infty} \left( \max\{s_j^l - s_{j-1}^l : j \in \{0, 1, 2, \ldots, k_l\}\} \right) = 0,
\]
then

\[
\lim_{l \rightarrow \infty} \sum_{j=1}^{k_l} \| f(s_j^l) - f(s_{j-1}^l) \| = W_b^a(f).
\]

Lemma 10 ([3], Ex. 68, P. 224). Let \( f : [a, b] \rightarrow X \) be a continuous function and let \( \{\{s_j^l\}_{j=0}^{k_l}\}_{l=1}^{\infty} \) be a normal sequence of partitions of the interval \([a, b]\). Then

\[
\lim_{l \rightarrow \infty} \sum_{j=1}^{k_l} \| \int_{s_{j-1}^l}^{s_j^l} f(t) dt \| = \int_a^b \| f(t) \| dt.
\]

3. The linear case. We shall consider the following initial value problem

\[
\begin{cases}
\frac{du}{dt} = Au + f(t) & t \in (0, T] \\
u(0) = x & x \in D,
\end{cases}
\]
where \( A : D \rightarrow X \) is the generator of a \( C_0 \) semigroup on a Banach space \( X \), and \( f : [0, T] \rightarrow X \) is a continuous function.

Definition 3. A continuous function \( u : [0, T] \rightarrow D \) is said to be a solution of the problem (2) if it is continuously differentiable in \((0, T]\) and satisfies the equation \( \frac{du}{dt}(t) = Au(t) + f(t) \), for \( t \in (0, T] \), and \( u(0) = x \).
Lemma 11 ([5], Th. 2.4, p. 107). Let \( A : D \rightarrow X \) be the generator of a \( C_0 \) semigroup \( \{ S(t) \} \) and let \( f : [0,T] \rightarrow X \) be a continuous function. We define
\[
v(t) = \int_0^t S(t-s)f(s)ds \quad \text{for} \quad t \in [0,T].
\]
If \( v(t) \in D \) for \( t \in (0,T] \), and the function \( t \rightarrow Av(t) \) is continuous on \( (0,T] \), then the problem (2) has a solution.

Remark 1. From the proof of Lemma 11 it follows that the solution of (2) has the form:
\[
u(t) = S(t)x + v(t) \quad \text{for} \quad t \in [0,T].
\]

Lemma 12. Let \( X \) be a reflexive Banach space and let \( A : D \rightarrow X \) be the generator of a \( C_0 \) semigroup \( \{ S(t) \} \) on \( X \). Let \( f : [0,T] \rightarrow X \) be a continuous function with bounded variation on \( [0,T] \). Then
\[
v(t,s) = \int_s^t S(t-r)f(r)dr \in D \quad \text{for} \quad 0 \leq s \leq t \leq T.
\]

We present here a proof similar to the proof of Lemma 1 in [1].
There exist constants \( M \geq 1, \omega > 0 \) such that \( \| S(t) \| \leq Me^{\omega t} \) for \( t \in [0,\infty) \)
(See [5] p. 4). Let \( t, s \in [0,T] \) and \( s \leq t \). For every \( n \in \{1,2,3,\ldots\} \) we write:
\[
u_n(t,s) = \int_s^t S(t-r)f_n(r)dr, \quad \text{where}
\]
\[
f_n(r) = \sum_{j=0}^{n-1} f(r_j)\chi_{B_j}(r)dr \quad \text{for} \quad r \in [s,t],
\]
\[
B_j = [r_j, r_{j+1}) \quad \text{for} \quad j \in \{0,1,2,\ldots,n-2\}, \quad B_{n-1} = [r_{n-1}, r_n],
\]
\[
r_j = s + \frac{t-s}{n} \cdot j \quad \text{for} \quad j \in \{0,1,2,\ldots,n\}, \quad \text{and} \quad \chi_{B_j}
\]
denotes the characteristic function of \( B_j \) for \( j \in \{0,1,2,\ldots,n-1\} \).
It is clear that \( \{f_n\} \) converges uniformly on \([s,t]\) to \( f \) and \( \{u_n(t,s)\} \) converges to \( v(t,s) \) when \( n \rightarrow \infty \).
\[ A u_n(t, s) = \sum_{j=0}^{n-1} [S(t - r_j)f(r_j) - S(t - r_{j+1})f(r_j)] \]
\[ = \sum_{j=0}^{n-1} S(t - r_j)f(r_j) - \sum_{j=1}^{n} S(t, r_j)f(r_{j-1}) \]
\[ = S(t - s)f(s) + \sum_{j=1}^{n-1} S(t - r_j)[f(r_j) - f(r_{j-1})] - S(0)f(r_{n-1}) \]
\[ = (S(t - s) - I)f(s) + \sum_{j=1}^{n-1} S(t - r_j)[f(r_j) - f(r_{j-1})] + [f(s) - f(r_{n-1})]. \]

Hence

(3) \[ \| A u_n(t, s) \| \leq \| (S(t - s) - I)f(s) \| + (Me^{\omega T} + 1)W_s^t(f). \]

Thus \( \{ A u_n(t, s) \} \) is uniformly bounded relative to \( n, t \) and \( s \).

On the other hand we have

\[ < A u_n(t, s), x^* > = < u_n(t, s), A^*x^* > \rightarrow < v(t, s), A^*x^* > \text{ for } x^* \in D(A^*). \]

By Lemma 1, \( D(A^*) \) is dense in \( X^* \), so by Lemma 5, \( \{ A u_n(t, s) \} \) is weakly convergent since \( \{ A u_n(t, s) \} \) is bounded (see (3)). From Lemma 2, \( \{ A u_n(t, s) \} \) converges weakly to an element of \( X \) denoted by \( y \) so by Lemma 3, \( v(t, s) \in D \) and \( A v(t, s) = y \) for \( s, t \in [0, T] \), \( s \leq t \).

REMARK 2. The inequality (3) leads to

(4) \[ \| A v(t, s) \| \leq \| (S(t - s) - I)f(s) \| + (Me^{\omega T} + 1)W_s^t(f). \]

LEMMA 13. Let \( X, A, S(t), f \) be as in Lemma 12. Then the function \( w: [0, T] \rightarrow X \) defined by \( w(t) = A v(t) \), where \( v(t) = v(t, 0) \), is continuous.

We present here a proof similar to the proof of Lemma 2 in [1]:

By Lemma 12 the function \( w \) is well defined. Let \( t \in [0, T] \) be fixed and take \( h > 0 \) such that \( t + h \in [0, T] \). We have

\[ w(t + h) - w(t) = A \int_0^{t+h} S(t + h - s)f(s)ds - A \int_0^t S(t - s)f(s)ds \]
\[ = A \int_0^t [S(t + h - s) - S(t - s)]f(s)ds + \int_t^{t+h} S(t + h - s)f(s)ds \]
\[ = (S(h) - I)w(t) + A \int_t^{t+h} S(t + h - s)f(s)ds. \]
From this and Remark 2 we have
\[
\| w(t + h) - w(t) \| \leq \| (S(h) - I)w(t) \| + \| (S(h) - I)f(t) \| \\
+ (Me^{\omega T} + 1)W_{t+h}(f).
\]
By the strong continuity of \( t \to S(t) \) the first and the second term of the right hand side of (5) tend to zero when \( h \searrow 0 \). By Lemma 8 the third term on the right hand side of (5) tends to 0 when \( h \searrow 0 \). Therefore
\[
\lim_{h \searrow 0} w(t + h) = w(t).
\]
The equality (6) proves that \( w \) is right continuous in \([0, T)\). To prove that \( w \) is left continuous in \((0, T]\) we define the family \( \{v_\varepsilon\}_{\varepsilon > 0} \) of functions \( v_\varepsilon : [0, T] \to X \) for each \( \varepsilon > 0 \) by
\[
v_\varepsilon(t) = \begin{cases} 
    \int_0^{t-\varepsilon} S(t-s)f(s)ds & \text{for } \varepsilon \leq t \leq T \\
    0 & \text{for } 0 \leq t < \varepsilon.
\end{cases}
\]
Since
\[
\int_0^{t-\varepsilon} S(t-s)f(s)ds = S(\varepsilon) \int_0^{t-\varepsilon} S(t-\varepsilon-s)f(s)ds = S(\varepsilon)v(t-\varepsilon) \text{ for } \varepsilon > 0
\]
we can write
\[
v_\varepsilon(t) = \begin{cases} 
    S(\varepsilon)v(t-\varepsilon) & \text{for } \varepsilon \leq t \leq T \\
    0 & \text{for } 0 \leq t < \varepsilon.
\end{cases}
\]
It follows from Lemma 12 that \( v(t-\varepsilon) \in D \) for \( t \in [\varepsilon, T] \). Therefore we can define a mapping \( w_\varepsilon : [0, T] \to X \) by the formula
\[
w_\varepsilon(t) = Av_\varepsilon(t), \quad t \in [0, T].
\]
The equality \( w_\varepsilon(t) = S(\varepsilon)w(t-\varepsilon) \) and (6) imply the right continuity of the function \( w_\varepsilon \) in \([0, T)\) for \( \varepsilon > 0 \). Now we shall prove that \( w_\varepsilon \) is left continuous in \((0, T]\) for \( \varepsilon > 0 \). Let \( h \in (0, \varepsilon) \) be such that \( t - h \in [\varepsilon, T] \) when \( t \in (\varepsilon, T] \) is fixed. We have:
\[
w_\varepsilon(t) - w_\varepsilon(t-h) = A \int_0^{t-\varepsilon} S(t-s)f(s)ds \\
-A \int_0^{t-h-\varepsilon} S(t-h-s)f(s)ds \\
= A \int_0^{t-\varepsilon} [S(t-s) - S(t-h-s)]f(s)ds + A \int_{t-\varepsilon-h}^{t-\varepsilon} S(t-h-s)f(s)ds \\
= [S(\varepsilon) - S(\varepsilon - h)]w(t-\varepsilon) + A \int_{t-\varepsilon-h}^{t-\varepsilon} S(t-h-s)f(s)ds.
\]
The first term on the right hand side of (7) tends to 0 when \( h \downarrow 0 \) and

\[
\| A \int_{t-h}^{t-\varepsilon} S(t-h-s)f(s)ds \| \leq \| S(\varepsilon-h) \| \| A \int_{t-h}^{t-\varepsilon} S(t-\varepsilon-s)f(s)ds \| \\
\leq Me^{\omega T} \| (S(h)-I)f(t-h-\varepsilon) \| + Me^{\omega T}(Me^{\omega T}+1)W_{t-h-\varepsilon}^{t-e}(f).
\]

Since the first term on the right hand side tends to 0 when \( h \downarrow 0 \) by Lemma 7 and the second term tends to 0 when \( h \downarrow 0 \) by Lemma 8, we have:

\[
\lim_{h \downarrow 0} w_{\varepsilon}(t-h) = w_{\varepsilon}(t) \quad \text{for} \quad t \in (\varepsilon,T].
\]

The equality (8) proves the left continuity of \( w_{\varepsilon} \) in \((0,T]\) because \( w_{\varepsilon}(t) = 0 \) for \( t \in (0, \varepsilon] \). Therefore \( w_{\varepsilon} \) is continuous in \((0,T]\) for each \( \varepsilon > 0 \).

Now let us observe that:

\[
\| w(t) - w_{\varepsilon}(t) \| = \| A \int_{t-\varepsilon}^{t} S(t-s)f(s)ds \| \\
\leq \| (S(\varepsilon)-I)f(t-\varepsilon) \| + (Me^{\omega T}+1)W_{t-\varepsilon}^{t}(f).
\]

By Lemma 7 and Lemma 8 this inequality implies that \( w_{\varepsilon} \) tends to \( w \) uniformly with respect to \( t \in [0,T] \) when \( \varepsilon \to 0 \). So \( w \) is a continuous function in the interval \((0,T]\).

**THEOREM 1.** Let \( X \) be a reflexive Banach space and let \( A : D \to X \) be the generator of a \( C_{0} \) semigroup \( \{S(t)\}_{t \geq 0} \). If \( f : [0,T] \to X \) is a continuous function with bounded variation, then the problem (2) has a unique solution given by the formula:

\[
u(t) = S(t)x + \int_{0}^{t} S(t-s)f(s)ds.
\]

**PROOF.** From Lemmas 11–13 and Remark 1 the formula (9) gives a solution of (2). If \( w \) is a solution of (2), then the function \( s \to S(t-s)w(s) \) is continuously differentiable in \((0,T]\) and

\[
\frac{d}{ds}(S(t-s)w(s)) = S(t-s)f(s).
\]

Integrating (10) from 0 to \( t \) we obtain

\[
w(t) = S(t)x + \int_{0}^{t} S(t-s)f(s)ds = u(t)
\]

and the uniqueness follows.
4. The nonlinear case.

DEFINITION 4. A continuous function \( u : [0, T] \rightarrow X \) is said to be a solution of the problem (1) if it is continuously differentiable in \((0, T)\),

\[
\frac{du}{dt}(t) = Au(t) + f(t, u(t)) \quad \text{for} \quad t \in (0, T], \quad \text{and} \quad u(0) = x.
\]

LEMMA 13 ([7] TH. 4.3). If \( u \) is a solution of the problem (1), then \( u \) is a solution of the following integral equation:

\[
(11) \quad u(t) = S(t)x + \int_0^t S(t-s)f(s, u(s))ds.
\]

PROOF. If \( u \) is a solution of the problem (1), then \( u(s) \in D \) for \( s \in (0, T] \) and \( \frac{d}{ds}(S(t-s)u(s)) = S(t-s)f(s, u(s)) \). By integration from 0 to \( t \) we obtain

\[
u(t) = S(t)x + \int_0^t S(t-s)f(s, u(s))ds.
\]

LEMMA 14 ([7] TH. 4.5). Let \( A : D \rightarrow X \) be the generator of a \( C_0 \) semigroup on \( X \). Let \( f : [0, T] \times X \rightarrow X \) be a continuous function satisfying a Lipschitz condition with respect to the second variable (i.e. there exists \( L > 0 \) such that for every \( y, z \in X \) and \( t \in [0, T] \)

\[
f(t, z) - f(t, y) \leq L \| z - y \|
\]

Then the equation (11) has a unique solution defined on \([0, T]\) which is continuous function.

LEMMA 15. If \( f : [a, b] \rightarrow X \) is a continuous function with bounded variation then for every \( \epsilon > 0 \) there exists \( \Delta > 0 \) such that

\[
\frac{1}{h} \int_a^{b-h} \| f(t+h) - f(t) \| dt \leq W_a^b(f) + \epsilon \quad \text{for} \quad h \in (0, \Delta)
\]

PROOF. Function \( f \) is uniformly continuous on \([a, b]\) hence there exists \( \Delta > 0 \) such that

\[
\| f(t+h) - f(t) \| \leq \epsilon \quad \text{for} \quad h \in [0, \Delta), \quad t \in [a, b-h].
\]

For fixed \( h \in (0, \Delta) \) there exists \( k = k(h) \) such that \( k \in N \) and \( a + kh \leq b - h < a + (k+1)h \).
We have

\begin{align*}
(12) \quad \frac{1}{h} \int_a^{b-h} \| f(t + h) - f(t) \| \, dt \\
= \frac{1}{h} \int_a^{a+kh} \| f(t + h) - f(t) \| \, dt + \frac{1}{h} \int_{a+kh}^{b-h} \| f(t + h) - f(t) \| \, dt \\
\leq \frac{1}{h} \int_a^{a+kh} \| f(t + h) - f(t) \| \, dt + \frac{1}{h} \cdot h \cdot \varepsilon \\
= \frac{1}{h} \int_a^{a+kh} \| f(t + h) - f(t) \| \, dt + \varepsilon
\end{align*}

It remains to show that

\[ \int_a^{a+kh} \frac{1}{h} \| f(t + h) - f(t) \| \, dt \leq W_a^b(f). \]

Let us define \( \{ \{t_j^m\}_{j=0}^{km} \}_{m=1}^{\infty} \) normal sequence of partitions of \([a, b]\) by

\[ t_j^m = a + j \frac{h}{m} \quad \text{for} \quad j \in \{0, 1, 2, \ldots, k \cdot m\}, \quad m \in \{1, 2, 3, \ldots\} \]

The integral sums for \( \int_a^{a+kh} \frac{1}{h} \| f(t + h) - f(t) \| \, dt \) has the form

\[ \sum_{j=0}^{km-1} \frac{1}{h} \| (f(a + j \frac{h}{m} + h) - f(a + j \frac{h}{m}) \| \frac{h}{m} \]

\[ = \sum_{r=0}^{m-1} \frac{1}{m} \sum_{p=0}^{k-1} \| (a + \frac{r h}{m} + (p + 1)h) - f(a + \frac{r h}{m}) \| \leq \frac{1}{m} \sum_{r=0}^{m-1} W_{a+\frac{r h}{m}}^{a+\frac{r h}{m}+k \cdot h}(f) \]

\[ \leq \frac{1}{m} \cdot m W_a^b(f) = W_a^b(f). \]

Hence

\begin{align*}
(13) \quad \frac{1}{h} \int_a^{a+kh} \| f(t + h) - f(t) \| \, dt \\
= \lim_{m \to \infty} \sum_{j=1}^{k \cdot m-1} \frac{1}{h} \| f(a + j \frac{h}{m} + h) - f(a + j \frac{h}{m}) \| \cdot \frac{h}{m} \leq W_a^b(f).
\end{align*}

From (12) and (13) follows that

\begin{align*}
(14) \quad \frac{1}{h} \int_a^{b-h} \| f(t + h) - f(t) \| \, dt \leq W_a^b(f) + \varepsilon \quad \text{for} \quad h \in (0, \Delta), \quad t \in [a, b - h].
\end{align*}
Lemma 16 ([3] Ex. 70 P. 225). If \( f: [a, b] \rightarrow X \) is a continuous function then

\[
W^b_a(f) = \lim_{h \to 0} \int_a^{b-h} \frac{\| f(t + h) - f(t) \|}{h} \, dt.
\]

We present the proof of the equality (15) for reader's convenience.

Let \( \{ \{ t^{(i)l}_{j=0} \}_{j=1}^{k_l+1} \}_{l=1}^{\infty} \) be a normal sequence of partitions of \([a, b]\) such that \( \{ t^{(i+1)l}_{j=0} \}_{j=0} \supset \{ t^{(i)l}_{j=0} \}_{j=0} \). We define \( s^l_j(h) = a + (t^l_j - a) \frac{b-h-a}{b-a} \) for \( h \in [0, b-a) \), \( j \in \{0, 1, 2, \ldots, k_l\} \). In this way we obtain \( \{ \{ s^{(i)l}_{j=0} \}_{l=1}^{\infty} \} \) the normal sequence of partitions of \([a, b-h]\) for \( h \in [0, b-a) \).

By Lemma 10 we have:

\[
\lim_{h \to 0} \int_a^{b-h} \frac{\| f(t + h) - f(t) \|}{h} \, dt
\]

\[
= \lim_{l \to \infty} \lim_{h \to 0} \frac{1}{h} \sum_{j=1}^{k_l} \| \int_{s^l_j}^{s^l_j+h} (f(t + h) - f(t)) \, dt \|
\]

\[
= \lim_{l \to \infty} \left[ \frac{1}{h} \sum_{j=1}^{k_l} \int_{s^l_j}^{s^l_j+h} f(t) \, dt - \int_{s^l_{j-1}}^{s^l_{j-1}+h} f(t) \, dt \right].
\]

If we prove that there exists

\[
\lim_{l \to \infty} \sum_{j=1}^{k_l} \frac{1}{h} \int_{s^l_j}^{s^l_j+h} f(t) \, dt - \int_{s^l_{j-1}}^{s^l_{j-1}+h} f(t) \, dt
\]

then by Lemma 9 we can write:

\[
\lim_{h \to 0} \frac{1}{h} \int_a^{b-h} \| f(t + h) - f(t) \| \, dt
\]

\[
= \lim_{l \to \infty} \sum_{j=1}^{k_l} \lim_{h \to 0} \frac{1}{h} \int_{s^l_j}^{s^l_j+h} f(t) \, dt - \int_{s^l_{j-1}}^{s^l_{j-1}+h} f(t) \, dt
\]

\[
= \lim_{l \to \infty} \sum_{j=1}^{k_l} \| f(t^l_j) - f(t^l_{j-1}) \| = W^b_a(f)
\]

because

\[
t^l_j = \lim_{h \to 0} s^l_j = \lim_{h \to 0} s^l_j(h), \text{ for } j \in \{0, 1, 2, \ldots, k_l\}, \; l \in \{1, 2, 3, \ldots\}.
\]
It remains to show the existence of the limit (17). Write
\[ \varphi(l, h) = \sum_{j=1}^{k_l} \frac{1}{h} \| \int_{s_j^l + h}^{s_j^l + h} f(t) dt - \int_{s_j^l}^{s_j^l - h} f(t) dt \| \]
for \( h > 0, l \in \{1, 2, 3, \ldots \} \) and
\[ \varphi(l, 0) = \lim_{h \to 0} \varphi(l, h) \quad \text{for} \quad l \in \{0, 1, 2, \ldots \}. \]

By Lemma 9 we have \( \varphi(l + 1, h) \geq \varphi(l, h) \) because \( \{t_j^{l+1}\} \supset \{t_j^l\} \) hence
\[ \sup\{\varphi(l, h) : l \in \{1, 2, 3, \ldots \}\} = \lim_{l \to \infty} \varphi(l, h) = \frac{1}{h} \int_a^b f(t + h) - f(t) dt. \]
If \( W_a^b(f) < \infty \), by Lemma 15
\[ \varphi(l, h) \leq \frac{1}{h} \int_a^b \| f(t + h) - f(t) \| dt \leq W_a^b(f) + \varepsilon \]
for all \( l \) and \( h \in (0, \Delta) \).

On the other hand, by Lemma 9 for every \( \varepsilon > 0 \) there exists \( l_0 \) such that
\[ \varphi(l, 0) > W_a^b(f) - \frac{\varepsilon}{2} \quad \text{for} \quad l > l_0. \]

There exists \( \delta > 0 \) such that
\[ \varphi(l_0, h) > \varphi(l_0, 0) - \frac{\varepsilon}{2} \quad \text{for} \quad h \in [0, \delta). \]

Since \( \varphi(l + 1, h) \geq \varphi(l, h) \) for \( h \in [0, b - a) \) and \( l \in \{1, 2, 3, \ldots \} \), from (19) and (20) we have
\[ \varphi(l, h) \geq \varphi(l_0, h) > \varphi(l_0, 0) - \frac{\varepsilon}{2} > W_a^b(f) - \varepsilon \quad \text{for} \quad l > l_0, \ h \in [0, \delta). \]

From (18) and (21) we have
\[ W_a^b(f) - \varepsilon < \varphi(l, h) < W_a^b(f) + \varepsilon \quad \text{for} \quad h < \min\{\Delta, \delta\}; \ l > l_0. \]

Hence the limit (17) exists and Lemma 16 is proved for \( f \) with bounded variation. If \( W_a^b(f) = \infty \), then for every \( C > 0 \) there exists \( l_0 \) such that
\[ \varphi(l, 0) > 2C \quad \text{for} \quad l > l_0. \]

By definition of \( \varphi(l, 0) \) there exists \( h_0 \) such that
\[ \varphi(l_0, 0) - C < \varphi(l_0, h) \quad \text{for} \quad h < h_0. \]

Since \( \varphi(l + 1, h) > \varphi(l, h) \) for \( h \in [0, b - a), \ l \in \{1, 2, 3, \ldots \} \), from (22), (23) we have
\[ \varphi(l, h) > \varphi(l_0, 0) - C > C \quad \text{for} \quad h \in [0, h_0), \ l > l_0. \]

This proves Lemma 16.
Lemma 17. Let \( A: D \to X \) be the generator of a \( C_0 \) semigroup on \( X \) and let \( f: [0, T] \times X \to X \) be a continuous function, satisfying the Lipschitz condition with respect to the second variable. Suppose that there exists \( h_0 > 0 \) such that

\[
\int_0^{T-h} \frac{\| f(t+h, u(t)) - f(t, u(t)) \|}{h} \, dt
\]

is bounded by \( K \) uniformly with respect to \( h \in (0, h_0) \), where \( u \) is the solution of the equation (11).

Then the function \( [0, T] \ni t \to f(t, u(t)) \in X \) has bounded variation.

Proof. We shall prove that \( u \) satisfies the Lipschitz condition, hence has bounded variation. Let \( h > 0 \) be such that \( t+h \in [0, T] \). We have

\[
\| u(t+h) - u(t) \| \leq \| S(t+h)x - S(t)x \| + \| \int_0^{t+h} S(t+h-s)f(s, u(s))ds - \int_0^t S(t-s)f(s, u(s))ds \| \\
\leq ah + \| \int_0^t S(t-s)[f(s+h, u(s+h)) - f(s+h, u(s)) + f(s+h, u(s)) - f(s, u(s))]ds \| \\
+hMe^{\omega T}m \leq (a + Me^{\omega T}m)h + LMMe^{\omega T} \int_0^t \| u(s+h) - u(s) \| \, ds \\
+hMe^{\omega T} \frac{1}{h} \int_0^t \| f(s+h, u(s)) - f(s, u(s)) \| \, ds \\
\leq (a + Me^{\omega T}m + Me^{\omega T} \cdot K) \cdot h + LMMe^{\omega T} \int_0^t \| u(s+h) - u(s) \| \, ds,
\]

where \( a \) is a Lipschitz constant for the function \( t \to S(t) \) strongly continuously differentiable by Lemma 4, \( L \) is a Lipschitz constant for \( f \) with respect to the second variable and \( m = \sup \{\| f(s, u(s)) \|, s \in [0, T] \} \).

By Lemma 6

\[
\| u(t+h) - u(t) \| \leq [(a + Me^{\omega T} \cdot m + Me^{\omega T} \cdot K) \exp(LMe^{\omega T}) \cdot h.
\]

Let us denote the Lipschitz constant for \( u \) by \( N \).
By Lemma 16 we have

\[
W_0^T(f(\cdot, u(\cdot))) = \lim_{h \to 0} \frac{1}{h} \int_0^{T-h} \| f(s + h, u(s + h)) - f(s, u(s)) \| \, ds
\leq \lim_{h \to 0} \left( \int_0^{T-h} \frac{1}{h} \| f(s + h, u(s + h)) - f(s + h, u(s)) \| \, ds + \int_0^{T-h} \frac{1}{h} \| f(s + h, u(s)) - f(s, u(s)) \| \, ds \right) \leq LNT + K.
\]

**Theorem 2.** Let \( X \) be a reflexive Banach space and let \( A : D \to X \) be the generator of a \( C_0 \) semigroup. Let \( f : [0, T] \times X \to X \) be a continuous function satisfying the Lipschitz condition with respect to the second variable. Suppose that there exists \( h_0 > 0 \) such that \( \frac{1}{h} \int_0^{T-h} \| f(s + h, u(s)) - f(s, u(s)) \| \, ds \) is uniformly bounded for \( h \in (0, h_0) \), where \( u \) is the solution of the equation (11). Then \( u \) is a unique solution of the problem (1).

**Proof.** The uniqueness of the solution of the problem (1) follows from Lemmas 13 and 14. By Lemma 17 the function \( t \to f(t, u(t)) \) satisfies the assumptions of Theorem 1 hence the problem:

\[
\begin{cases}
\frac{dw}{dt} = Aw + f(t, u(t)) & t \in (0, T] \\
w(0) = x & x \in D
\end{cases}
\tag{24}
\]

has a unique solution given by the formula

\[
w(t) = S(t)x + \int_0^t S(t - s)f(s, u(s))ds.
\]

But \( u \) satisfies (11) hence \( u = w \) is a solution of the problem (24). We have

\[
\frac{du}{dt}(t) = Au(t) + f(t, u(t)) \quad \text{for} \quad t \in (0, T] \quad \text{and} \quad u(0) = x
\]

so \( u \) is the solution of the problem (1).
References


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