ON THE CHAPLYGHIN METHOD
FOR PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS
OF THE FIRST ORDER

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Abstract. We consider the mixed problem for the semilinear partial differential-functional equation of the first order

\[ D_{x}z(x, y) = F(x, y, z(x, y)) + \sum_{i=1}^{n} f_{i}(x, y)D_{y_{i}}z(x, y), \]

\[(x, y) \in [0, a] \times [-b, b],\]

\[z(x, y) = \phi(x, y), \quad (x, y) \in [-\tau, a] \times [-b, b + \delta] \setminus (0, a] \times [-b, b),\]

where \(z_{(x, y)} : [-\tau, 0] \times [0, h] \rightarrow \mathbb{R}\) is a function defined by \(z_{(x, y)}(t, s) = z(x + t, y + s), (t, s) \in [-\tau, 0] \times [0, h]\). Using the method of characteristics and the method of differential-functional inequalities we prove, under suitable assumptions, a theorem on the convergence of the Chaplygin sequences to the solution of the problem (1), (2).

1. Introduction. If \(X, Y\) are any metric spaces then we denote by \(C(X; Y)\) the class of all continuous functions from \(X\) to \(Y\). Let \(B = [-\tau, 0] \times [0, h]\), where \(h = (h_{1}, \ldots, h_{n}) \in \mathbb{R}_{+}^{n}, \tau \in \mathbb{R}_{+}, (\mathbb{R}_{+} = [0, +\infty))\). For a given function \(z : [-\tau, a] \times [-b, b + \delta] \rightarrow \mathbb{R}\), where \(a > 0, b = (b_{1}, \ldots, b_{n}), b_{i} > 0, i = 1, \ldots, n\), and a point \((x, y) = (x, y_{1}, \ldots, y_{n}) \in [0, a] \times [-b, b],\) we consider the function \(z_{(x, y)} : B \rightarrow \mathbb{R}\) defined by

\[z_{(x, y)}(t, s) = z(x + t, y + s), \quad (t, s) \in B.\]

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For any $\bar{a} \in (0, a]$ we define the sets

$$
E_0^* = [-\tau, 0] \times [-b, b + h], \quad \partial_0 E_{\bar{a}} = [0, \bar{a}] \times [-b, b + h] \setminus (0, \bar{a}] \times [-b, b),$$
$$E_{\bar{a}} = [0, \bar{a}] \times [-b, b], \quad E_{\bar{a}}^* = E_0^* \cup \partial_0 E_{\bar{a}} \cup E_{\bar{a}}.
$$

For given functions $\phi: E_{\bar{a}}^* \cup \partial_0 E_a \to \mathbb{R}$, $F: E_a \times C(B; \mathbb{R}) \to \mathbb{R}$, $f = (f_1, \ldots, f_n): E_a \to \mathbb{R}^n$, we consider the following mixed problem

\begin{align*}
(1) \quad &D_x z(x, y) = F(x, y, z(x, y)) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} z(x, y) \\
&\text{for } (x, y) \in E_a,
(2) \quad &z(x, y) = \phi(x, y) \quad \text{for } (x, y) \in E_{\bar{a}}^* \cup \partial_0 E_a.
\end{align*}

We consider classical solutions of (1),(2) local with respect to the first variable. In other words, a function $z \in C(E_{\bar{a}}^*; \mathbb{R})$, where $\bar{a} \in (0, a]$, is said to be a solution of (1),(2) if it is of class $C^1$ on $E_{\bar{a}}$, satisfies equation (1) on $E_{\bar{a}}$ and fulfills the initial-boundary condition (2) on $E_{\bar{a}}^* \cup \partial_0 E_{\bar{a}}$.

In this paper we give sufficient conditions for the existence of two monotone sequences $\{u^{(m)}\}, \{v^{(m)}\}$ such that if $z$ is a solution of (1), (2) then $u^{(m)} \leq z \leq v^{(m)}$ on $E_{\bar{a}}$ and $\{u^{(m)}\}, \{v^{(m)}\}$ are uniformly convergent to $z$ on $E_{\bar{a}}$. The convergence that we get is of the Newton type, which means that

$$
0 \leq z(x, y) - u^{(m)}(x, y) \leq \frac{2A}{2^m}, \quad 0 \leq v^{(m)}(x, y) - z(x, y) \leq \frac{2A}{2^m}, \quad \text{on } E_{\bar{a}},
$$

where $A$ is some constant not dependent on $m$. The functions $u^{(m)}, v^{(m)}$ are defined as solutions of some linear differential-functional equations obtained by the linearization of (1). This method of approximating solutions of differential equations was introduced by Chaplygin in [2]. In the original Chaplygin method only one approximating sequence was defined (cf. [5], [8]). This method was applied by Mlak and Schechter [6] to the system of the first order semilinear partial differential equations and was extended by Nowotarska [7] to the case of an infinite system.

Note that because the model of functional dependence in (1) is based on the use of the operator $z_{(x,y)}$, the given function $F$ is a functional operator on $C(B; \mathbb{R})$ with respect to the last variable. The Chaplygin method for partial differential-functional equations with another model of functional dependence was considered by Kamont in [4] but linearization only with respect to the non-functional argument was allowed there.

We prove in this paper that Chaplygin sequences are monotone using the theorem on differential-functional inequalities given by Brandi, Kamont and
Salvadori [1] slightly modified for our model of functional dependence. The existence of these sequences follows from the theorem on existence and uniqueness of the initial-boundary problems which was given in [3]. We also give the estimate of the difference between the exact solution of (1), (2) and its approximations using the comparison theorem for first order partial differential-functional equations [1]. This estimate is of the same order as in the non-functional problem of [6] but better than that of [4]. This is due to the fact that in our method, unlike the method of [4], the linearization concerns the functional argument.

2. Differential-functional inequalities and a comparison theorem.
We now state two theorems which will be used to get the convergence of the Chaplygin sequence. Let $\Omega_a = E_a \times R \times C(B; R) \times R^n$.

Assumption $H_1$. Suppose that $\sigma \in C([0, a] \times R_+ \times C([-\tau, 0]; R_+); R_+)$ is such that

(i) $\sigma(t, 0, 0) = 0$ for $t \in [0, a]$;

(ii) $\bar{\omega}(t) = 0$ for $t \in [-\tau, a]$ is the unique solution of the problem

$$
\omega'(t) = \sigma(t, \omega(t), \omega_t), \quad t \in [0, a],
$$

$$
\omega(t) = 0, \quad t \in [-\tau, 0],
$$

where $\omega_t : [-\tau, 0] \rightarrow R_+$ is the function defined for any $t \in [0, a]$ by the formula $\omega_t(s) = \omega(t + s), \ t \in [-\tau, 0]$.

Theorem 1. Suppose that Assumption $H_1$ is satisfied,

$1^o$ $\mathcal{F} : \Omega_a \rightarrow R$ is a function of the variables $(x, y, p, w, q)$, nondecreasing with respect to the functional argument $w$ and such that for all $w, \bar{w} \in C(B; R)$ and $p, \bar{p} \in R$, where $\bar{p} \geq p$ and $\bar{w} \geq w$ on $E_a$, we have

$$
\mathcal{F}(x, y, \bar{p}, \bar{w}, q) - \mathcal{F}(x, y, p, w, q) \leq \sigma(x, \bar{p} - p, \max_{s \in [0, \bar{a}]} [\bar{w}(\cdot, s) - w(\cdot, s)])
$$

for $(x, y, q) \in E_a \times R^n$;

$2^o$ the derivative $D_q \mathcal{F} = (D_{q_1} \mathcal{F}, \ldots, D_{q_n} \mathcal{F})$ exists and fulfills the inequality $D_q \mathcal{F}(x, y, p, w, q) \geq 0$ on $\Omega_a$;

$3^o$ $u, v \in C(E_0^*; R)$, where $\bar{a} \in (0, a]$, are two functions such that the derivatives $D_x u, D_x v, D_y u, D_y v$ exist on $(0, \bar{a}] \times [-b, b] \setminus \partial_0 E_\bar{a}$ and the following initial-boundary inequality holds

$$
u(x, y) \leq v(x, y) \quad \text{for} \ (x, y) \in E_0^* \cup \partial_0 E_\bar{a};
$$

$4^o$ for any $(x, y) \in T$, where

$$
T = \{(x, y) \in (0, \bar{a}] \times [-b, b] \setminus \partial_0 E_\bar{a}; \ u(x, y) > v(x, y)\},
$$
we have the following differential-functional inequalities

\[ D_x u(x, y) \leq F(x, y, u(x, y), u(x, y), D_y u(x, y)), \]
\[ D_x v(x, y) \geq F(x, y, v(x, y), v(x, y), D_y v(x, y)). \]

Then we have

\[ u(x, y) \leq v(x, y) \quad \text{for} \quad (x, y) \in E_a^*. \]

We omit the proof of this theorem which is similar to that given in [1] for inequalities with another model of functional dependence.

**Assumption H\textsubscript{2}.** Suppose that

1° \( f = (f_1, \ldots , f_n) : E_a \rightarrow \mathbb{R}^n \) is a function of the variables \((x, y)\) and \( f(x, y) \geq 0 \) on \( E_a; \)

2° \( \sigma \in C([0, a] \times \mathbb{R}_+ \times C([-\tau, 0]; \mathbb{R}_+); \mathbb{R}_+) \) is nondecreasing with respect to the functional argument and for each \( \eta \in C([-\tau, 0]; \mathbb{R}_+) \) there is a right-hand maximum solution \( \omega_\eta \) of the problem

\[ \omega'(t) = \sigma(t, \omega(t), \omega_t), \quad t \in [0, a], \]
\[ \omega(t) = \eta(t), \quad t \in [-\tau, 0]. \]

**Theorem 2.** Suppose that Assumption \( H_2 \) is satisfied,

1° \( u \in C(E_a^*; \mathbb{R}) \), where \( a \in (0, a] \), is a function such that the derivatives \( D_x u, D_y u \) exist on \( (0, \bar{a}] \times [-b, b] \setminus \partial_0 E_{\bar{a}} \) and there exists a nondecreasing function \( \eta \in C([-\tau, 0]; \mathbb{R}_+) \) such that

\[ |u(x, y)| \leq \eta(x) \quad \text{for} \quad (x, y) \in E_0^*, \]

and

\[ |u(x, y)| \leq \eta(0) \quad \text{for} \quad (x, y) \in \partial_0 E_{\bar{a}}, \]

2° for any \( (x, y) \in (0, \bar{a}] \times [-b, b] \setminus \partial_0 E_{\bar{a}} \) the following differential-functional inequality holds

\[ \left| D_x u(x, y) - \sum_{i=1}^{n} f_i(x, y) D_y u(x, y) \right| \leq \sigma\left(x, |u(x, y)|, \left( \max_{s \in [-b, b+h]} |u(\cdot, s)| \right)_x \right), \]

where \( \left( \max_{s \in [-b, b+h]} |u(\cdot, s)| \right)_x(t) \) denotes the function on \([-\tau, 0]\) defined by

\[ \left( \max_{s \in [-b, b+h]} |u(\cdot, s)| \right)_x(t) = \max_{s \in [-b, b+h]} |u(x + t, s)|. \]

Then we have

\[ |u(x, y)| \leq \omega_\eta(x) \quad \text{for} \quad (x, y) \in E_{\bar{a}}^*, \]

where \( \omega_\eta \) is the right-hand maximum solution of (3).

We omit the proof of this theorem which is similar to that given in [1].
**Assumption H₃.** Suppose that
\[ f = (f₁, \ldots, fₙ) : Eₐ \rightarrow \mathbb{R}^n \] is a function of the variables \((x, y)\) and \(f(x, y) \geq 0\) on \(Eₐ\);\n\[ F \in C(Eₐ \times C(B; \mathbb{R}); \mathbb{R}) \] is a function of the variables \((x, y, w)\) such that the derivative \(D_wF(x, y, w)\) exists in \(Eₐ \times C(B; \mathbb{R})\), and for all \((x, y) \in Eₐ, w, \bar{w} \in C(B; \mathbb{R})\), we have
\[
(i) \quad D_wF(x, y, w) \circ h \geq 0 \quad \text{if} \quad h \in C(B; \mathbb{R}_+);
(ii) \quad D_wF(x, y, \bar{w}) \circ h \geq D_wF(x, y, w) \circ h \quad \text{if} \quad h \in C(B; \mathbb{R}_+), \quad \bar{w} \geq w,
\]
where \(D_wF(x, y, w) \circ h\) denotes the composite function of \(h\) and the linear operator \(D_wF(x, y, w)\).

**Assumption H₄.** Suppose that
\[ \text{i° we have two functions } u, v \in C(Eₐ^*; \mathbb{R}), \text{ where } \bar{a} \in (0, a], \text{ such that} \]
\[
\begin{align*}
D_xu(x, y) &\leq F(x, y, u(x, y)) + \sum_{i=1}^{n} f_i(x, y)D_yu(x, y), \\
D_xv(x, y) &\geq F(x, y, v(x, y)) + \sum_{i=1}^{n} f_i(x, y)D_yv(x, y),
\end{align*}
\]
on \(Eₐ,\) and
\[ u(x, y) \leq \phi(x, y) \leq v(x, y) \quad \text{on} \quad Eₐ^* \cup \partial₀Eₐ; \]
\[ \text{ii° we have two initial-boundary functions } \alpha, \beta \in C(Eₐ^* \cup \partial₀Eₐ; \mathbb{R}) \text{ satisfying the inequalities} \]
\[
\begin{align*}
u(x, y) &\leq \alpha(x, y) \leq \phi(x, y) \leq \beta(x, y) \leq v(x, y) \quad \text{on} \quad Eₐ^* \cup \partial₀Eₐ.
\end{align*}
\]
Given \(u, v\) satisfying Assumption H₄ we define the functions \(G(\cdot; u), H(\cdot; u, v) : Eₐ \times C(B; \mathbb{R}) \times \mathbb{R}^n \rightarrow \mathbb{R}\)
by
\[
\begin{align*}
G(x, y, w, q; u) &= F(x, y, u(x, y)) \\
&\quad + D_wF(x, y, u(x, y)) \circ (w - u(x, y)) + \sum_{i=1}^{n} f_i(x, y)q_i,
\end{align*}
\]
\[
\begin{align*}
H(x, y, w, q; u, v) &= F(x, y, v(x, y)) \\
&\quad + D_wF(x, y, u(x, y)) \circ (w - v(x, y)) + \sum_{i=1}^{n} f_i(x, y)q_i,
\end{align*}
\]
for \((x, y, w, q) \in Eₐ \times C(B; \mathbb{R}) \times \mathbb{R}^n\).

Now, we prove a theorem which is essential in the proof of the monotonicity of Chaplygin sequences.
Theorem 3. If Assumptions $H_3, H_4$ are satisfied, $z^*$ is a solution of (1), (2) and $U, V$ are solutions of the problems

(7) \[ D_x z(x, y) = G(x, y, z(x, y), D_y z(x, y); u) \quad \text{on } E_a, \]
\[ z(x, x) = \alpha(x, y) \quad \text{on } E_0^* \cup \partial_0 E_a, \]

and

(8) \[ D_x z(x, y) = \mathcal{H}(x, y, z(x, y), D_y z(x, y); u, v) \quad \text{on } E_a, \]
\[ z(x, x) = \beta(x, y) \quad \text{on } E_0^* \cup \partial_0 E_a, \]

respectively, then

(9) \[ u(x, y) \leq U(x, y) \leq z^*(x, y) \leq V(x, y) \leq v(x, y) \quad \text{on } E_a^* \]

and the following differential-functional inequalities

(10) \[ D_x U(x, y) \leq F(x, y, U(x, y)) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} U(x, y), \]

(11) \[ D_x V(x, y) \geq F(x, y, V(x, y)) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} V(x, y), \]

hold true on $E_a^*$.

Proof. We first prove that $u \leq U$ on $E_a^*$. The differential-functional inequality (4) may be written in the form

\[ D_x u(x, y) \leq F(x, y, u(x, y)) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} u(x, y) \]
\[ = G(x, y, u(x, y), D_y u(x, y); u), \]

where $(x, y) \in E_a$, and since $U$ is a solution of (7), we have our claim by (6) and Theorem 1. Our next goal is to prove that $U \leq z^*$ on $E_a^*$. For any $(x, y) \in E_a$ we have

\[ F(x, y, U(x, y)) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} U(x, y) - D_x U(x, y) \]
\[ = F(x, y, U(x, y)) - F(x, y, u(x, y)) - D_w F(x, y, u(x, y)) \circ (U(x, y) - u(x, y)) \]
\[ = D_w F(x, y, u(x, y)) + \theta(U(x, y) - u(x, y)) \circ (U(x, y) - u(x, y)) \]
\[ - D_w F(x, y, u(x, y)) \circ (U(x, y) - u(x, y)) \geq 0, \]
where \( \theta \in (0,1) \), and thus we get (10). The last inequality in the above estimate follows from Assumption H\(_3\) and from the already proved relation \( U \geq u \). Since \( z^* \) is a solution of (1), (2) we get our claim from Theorem 1. Analogously, since \( V \) is a solution of (8) and (5) may be written in the form
\[
D_x v(x,y) \leq F(x,y, u(x,y)) + \sum_{i=1}^n f_i(x,y) D_{y_i} v(x,y)
\]
\[
= \mathcal{H}(x,y, u(x,y), D_y v(x,y); u,v),
\]
where \( (x,y) \in E_{\bar{a}} \), we have \( V \leq v \) on \( E_{\bar{a}}^* \) by (6) and Theorem 1.

Finally in order to get \( z^* \leq V \) on \( E_{\bar{a}}^* \) we need an auxiliary inequality \( U \leq V \) on \( E_{\bar{a}}^* \). The latter inequality follows from the fact that \( V \) is a solution of (8) and the estimate
\[
\mathcal{H}(x,y, U(x,y), D_y U(x,y); u,v) - D_x U(x,y)
\]
\[
= F(x,y, u(x,y)) + D_w F(x,y, u(x,y)) o (U(x,y) - u(x,y))
\]
\[
- F(x,y, u(x,y)) - D_w F(x,y, u(x,y)) o (U(x,y) - u(x,y))
\]
\[
= D_w F(x,y, u(x,y)) + \theta (V(x,y) - u(x,y)) o (V(x,y) - u(x,y))
\]
\[
- D_w F(x,y, u(x,y)) o (V(x,y) - u(x,y)) \geq 0,
\]
by (6) and Theorem 1. With the help of this inequality we have
\[
V(x,y) + \theta (V(x,y) - u(x,y)) \geq V(x,y) \geq U(x,y) \geq u(x,y),
\]
for any \( (x,y) \in E_{\bar{a}} \), \( \theta \in (0,1) \) and, consequently, by the already proved relation \( v \geq V \) and Assumption H\(_3\) we get
\[
F(x,y, V(x,y)) + \sum_{i=1}^n f_i(x,y) D_{y_i} V(x,y) - D_x V(x,y)
\]
\[
= F(x,y, V(x,y)) - F(x,y, u(x,y)) - D_w F(x,y, u(x,y)) o (V(x,y) - u(x,y))
\]
\[
= D_w F(x,y, u(x,y)) + \theta (V(x,y) - u(x,y)) o (V(x,y) - u(x,y))
\]
\[
- D_w F(x,y, u(x,y)) o (V(x,y) - u(x,y)) \leq 0,
\]
for \( (x,y) \in E_{\bar{a}} \), which is (11). Since \( z^* \) is a solution of (1), we get our final claim using (6) and Theorem 1 once again.

3. Existence of solutions of a semilinear problem. Let us consider the following semilinear initial-boundary problem
\[
D_x z(x,y) = G(x,y, z(x,y)) + \sum_{i=1}^n f_i(x,y) D_{y_i} z(x,y)
\]
for \( (x,y) \in E_{\bar{a}}, \)
\[
z(x,y) = \phi(x,y) \quad \text{for} \ (x,y) \in E_{0}^* \cup \partial_0 E_{\bar{a}},
\]
where \( G : E_{\bar{a}} \times C(B; \mathbb{R}) \rightarrow \mathbb{R} \) is a given function and \( f, \phi \) are as in (1).
Assumption $H_5$. Suppose that $\phi \in C^1(E_0^* \cup \partial_0E_\alpha; \mathbb{R})$ and there are constants $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathbb{R}_+$ such that we have

$$|\phi(x,y)| \leq \Lambda_0, \quad |D_x\phi(x,y)| \leq \Lambda_1, \quad |D_y\phi(x,y)| \leq \Lambda_1,$$

$$|D_x\phi(x,y) - D_x\phi(\bar{x}, \bar{y})| \leq \Lambda_2[|x - \bar{x}| + |y - \bar{y}|],$$

$$|D_y\phi(x,y) - D_y\phi(\bar{x}, \bar{y})| \leq \Lambda_2[|x - \bar{x}| + |y - \bar{y}|].$$

Suppose that Assumption $H_5$ is satisfied and we are given $Q_0, Q_1, Q_2 \in \mathbb{R}_+$ such that $Q_i \geq \Lambda_i$, $i = 0, 1, 2$, we will denote by $C^{1,L}_a(Q)$, where $a \in (0, \alpha]$, the set of all functions $z \in C(E_\alpha; \mathbb{R})$ such that

(i) $\quad z(x,y) = \phi(x,y)$, on $E_0^* \cup \partial_0E_\alpha$

and

(ii) $|z(x,y)| \leq Q_0, \quad |D_xz(x,y)| \leq Q_1, \quad |D_yz(x,y)| \leq Q_1,$

$$|D_xz(x,y) - D_xz(\bar{x}, \bar{y})| \leq Q_2[|x - \bar{x}| + |y - \bar{y}|],$$

$$|D_yz(x,y) - D_yz(\bar{x}, \bar{y})| \leq Q_2[|x - \bar{x}| + |y - \bar{y}|],$$

on $E_\alpha^*$.

Assumption $H_6$. Suppose that

1° $f = (f_1, \ldots, f_n) \in C(E_\alpha; \mathbb{R}^n)$ is a function of the variables $(x,y)$ and there exists the derivative $D_yf$ on $E_\alpha$;

2° there exist constants $L_0, L_1, L_2 \geq 0$ such that

$$|f(x,y)| \leq L_0, \quad |f(x,y) - f(\bar{x}, \bar{y})| \leq L_1|x - \bar{x}|,$$

$$|D_yf(x,y)| \leq L_1, \quad |D_yf(x,y) - D_yf(\bar{x}, \bar{y})| \leq L_2[|x - \bar{x}| + |y - \bar{y}|],$$

3° $f(x,y) \geq 0$ on $E_\alpha$ and there is $\delta_0 > 0$ such that

$$f_i(x,y) \geq \delta_0 \quad \text{for } i = 1, \ldots, n, \text{ and } (x,y) \in E_\alpha \text{ such that } y_i = b_i.$$

If $\| \cdot \|_0$ denotes the supremum norm in $C(B; Y)$, where $Y$ is an Euclidean space, then the norm in $C^1(B; Y)$ is defined by $\|w\|_1 = \|w\|_0 + \|D_{xy}w\|_0$, where $D_{xy}w$ denotes the Jacobi matrix of $w$.

For any $w \in C(B; Y)$ let

$$\|w\|_L = \sup\{|w(x,y) - w(\bar{x}, \bar{y})| \cdot [|x - \bar{x}| + |y - \bar{y}|]^{-1}: (x,y), (\bar{x}, \bar{y}) \in B\}.$$ 

Then if we put $\|w\|_{0,L} = \|w\|_0 + \|w\|_L$, $\|w\|_{1,L} = \|w\|_1 + \|D_{xy}w\|_L$, then we denote by $C^{i,L}(B; Y)$, $i = 0, 1$, the space of all functions $z \in C^i(B; Y)$ such that $\|z\|_{i,L} < +\infty$ with the norm $\| \cdot \|_{i,L}$.
ASSUMPTION $H_7$. Suppose that

1° $G \in C(E_a \times C(B; \mathbb{R}); \mathbb{R})$ is a function of the variables $(x, y, w)$ and there exist derivatives $D_yG, D_wG$ on $E_a \times C(B; \mathbb{R})$;

2° there exist nondecreasing functions $M_0, M_1, M_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $(x, y), (\bar{x}, \bar{y}) \in E_a$ we have

$$|G(x, y, w)| \leq M_0(q), \quad w \in C(B; \mathbb{R}), \quad \|w\|_0 \leq q,$$

$$|G(x, y, w) - G(\bar{x}, y, w)| \leq M_1(q)|x - \bar{x}|, \quad w \in C^{0,L}(B; \mathbb{R}), \quad \|w\|_{0,L} \leq q,$$

$$|D_yG(x, y, w)| \leq M_1(q), \quad \|D_wG(x, y, w)\| \leq M_1(q),$$

$$w \in C^1(B; \mathbb{R}), \quad \|w\|_1 \leq q,$$

and

$$|D_yG(x, y, w) - D_yG(x, y, w)| \leq M_2(q)[|x - \bar{x}| + |y - \bar{y}| + \|w - \bar{w}\|_0],$$

$$\|D_wG(x, y, w) - D_wG(x, y, w)\| \leq M_2(q)[|x - \bar{x}| + |y - \bar{y}| + \|w - \bar{w}\|_0],$$

where $w, \bar{w} \in C^{1,L}(B; \mathbb{R}), \quad \|w\|_{1,L}, \|\bar{w}\|_{1,L} \leq q$;

3° the following consistency condition

$$D_x\phi(x, y) - \sum_{i=1}^n f_i(x, y)D_y\phi(x, y) = G(x, y, \phi(x, y)),$$

holds true on $(E^*_0 \cup \partial_0E_a) \cap E_a$.

Now, we state a theorem on existence and uniqueness of the local solutions of (12), (13) which ensures the existence of Chaplygin sequences.

**THEOREM 4.** If Assumptions $H_5$-$H_7$ are satisfied, then we may choose $Q_0, Q_1, Q_2, Q_i > \Lambda_i, \quad i = 0, 1, 2$, such that the problem (12), (13) has a unique solution on $E_a$ for sufficiently small $\bar{a} \in (0, a]$ in the class $C^{1,L}(Q)$.

Theorem 4 follows from Theorem 2 of [3] where not only $G$ but also $f$ may depend on the functional argument. The proof of that theorem is based on the bicharacteristics method and on the Banach fixed-point theorem.

4. Chaplygin sequences. We are now able to define two monotone Chaplygin sequences.

ASSUMPTION $H_8$. Suppose that

1° $F \in C(E_a \times C(B; \mathbb{R}); \mathbb{R})$ and the derivatives $D_yF, D_wF, D_{yw}F, D_{ww}F$ exist on $E_a \times C(B; \mathbb{R})$;

2° for all $(x, y) \in E_a, w, \bar{w} \in C(B; \mathbb{R})$, we have

(i) \hspace{1cm} $D_wF(x, y, w) \circ h \geq 0$ if $h \in C(B; \mathbb{R}_+)$;

(ii) \hspace{1cm} $D_wF(x, y, \bar{w}) \circ h \geq D_wF(x, y, w) \circ h$ if $h \in C(B; \mathbb{R}_+), \bar{w} \geq w$. 


there are constants $K_0, K_1, K_2, K_3 \in \mathbb{R}_+$ such that

$$
|F(x, y, w)| \leq K_0, \quad |F(x, y, w) - F(\bar{x}, y, w)| \leq K_1|x - \bar{x}|,
$$

$$
|D_y F(x, y, w)| \leq K_1, \quad \|D_w F(x, y, w)\| \leq K_1,
$$

$$
\|D_{yw} F(x, y, w)\| \leq K_2, \quad \|D_{ww} F(x, y, w)\| \leq K_2,
$$

$$
\|D_w F(x, y, w) - D_w F(\bar{x}, y, w)\| \leq K_2|x - \bar{x}|,
$$

$$
|D_y F(x, y, w) - D_y F(\bar{x}, y, w)| \leq K_2|x - \bar{x}|,
$$

$$
\|D_{yw} F(x, y, w) - D_{yw} F(\bar{x}, \bar{y}, \bar{w})\| \leq K_3[|x - \bar{x}| + |y - \bar{y}| + \|w - \bar{w}\|_0],
$$

$$
\|D_{ww} F(x, y, w) - D_{ww} F(\bar{x}, \bar{y}, \bar{w})\| \leq K_3[|x - \bar{x}| + |y - \bar{y}| + \|w - \bar{w}\|_0],
$$

for $(x, y), (\bar{x}, \bar{y}) \in E_0, w, \bar{w} \in C(B; \mathbb{R})$.

**Assumption H9.** Suppose that

1° there are two sequences $\{\alpha^{(m)}\}, \{\beta^{(m)}\}$ of initial-boundary functions fulfilling Assumption H5 such that the inequalities

$$
\alpha^{(m)}(x, y) \leq \alpha^{(m+1)}(x, y) \leq \phi(x, y) \leq \beta^{(m+1)}(x, y) \leq \beta^{(m)}(x, y),
$$

hold for $(x, y) \in E^*_0 \cup \partial_0 \bar{E}_a$, $m \in \mathbb{N}$;

2° the following consistency conditions

$$
D_x \alpha^{(m)}(x, y) - \sum_{i=1}^{n} f_i(x, y) D_{y_i} \alpha^{(m)}(x, y)
$$

$$
= F(x, y, \alpha^{(m)}(x, y)) + D_w F(x, y, \alpha(x, y)) \circ (\alpha^{(m+1)}(x, y) - \alpha^{(m)}(x, y)),
$$

$$
D_x \beta^{(m)}(x, y) - \sum_{i=1}^{n} f_i(x, y) D_{y_i} \beta^{(m)}(x, y)
$$

$$
= F(x, y, \beta^{(m)}(x, y)) + D_w F(x, y, \alpha(x, y)) \circ (\beta^{(m+1)}(x, y) - \beta^{(m)}(x, y)),
$$

hold true for $(x, y) \in (E^*_0 \cup \partial_0 \bar{E}_a) \cap \bar{E}_a$, $m \in \mathbb{N}$;

3° there exist functions $u^{(0)}, v^{(0)} \in C(E^*_a; \mathbb{R})$ such that

$$
D_x u^{(0)}(x, y) \leq F(x, y, u^{(0)}_{(x,y)}) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} u^{(0)}(x, y),
$$

$$
D_x v^{(0)}(x, y) \geq F(x, y, v^{(0)}_{(x,y)}) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} v^{(0)}(x, y),
$$

on $E_0$, and

$$
u^{(0)}(x, y) = \alpha^{(0)}(x, y), \quad v^{(0)}(x, y) = \beta^{(0)}(x, y), \quad \text{on } E^*_0 \cup \partial_0 \bar{E}_a.$$
Let $T_{\alpha \beta} : C(E^{*}_a; \mathbb{R}) \times C(E^{*}_a; \mathbb{R}) \rightarrow C(E^{*}_a; \mathbb{R}) \times C(E^{*}_a; \mathbb{R})$, $\bar{a} \in (0, a]$, be the operator defined by $T_{\alpha \beta}[u, v] = [U, V]$, where $U, V$ are solutions of the problems (7),(8), respectively. We consider the sequences $\{u^{(m)}\}, \{v^{(m)}\}$ defined as follows:

1° let $u^{(0)}, v^{(0)} \in C(E^{*}_a; \mathbb{R})$ be two functions satisfying condition 3° of Assumption $H_9$;

2° if $u^{(m)}, v^{(m)} \in C(E^{*}_a; \mathbb{R})$ are already defined such that

$$D_x u^{(m)}(x, y) \leq F(x, y, u^{(m)}(x, y)) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} u^{(m)}(x, y),$$

$$D_x v^{(m)}(x, y) \geq F(x, y, v^{(m)}(x, y)) + \sum_{i=1}^{n} f_i(x, y) D_{y_i} v^{(m)}(x, y),$$

on $E_{\bar{a}}$, then

$$[u^{(m+1)}, v^{(m+1)}] = T_{\alpha^{(m+1)} \beta^{(m+1)}}[u^{(m)}, v^{(m)}], \quad m \in \mathbb{N}. $$

If Assumptions $H_5, H_6, H_8, H_9$ are satisfied then it follows by Theorem 4 that the sequences $\{u^{(m)}\}, \{v^{(m)}\}$ are well defined and by Theorem 3 we immediately see that the following inequalities

$$(14) \quad u^{(m)}(x, y) \leq u^{(m+1)}(x, y) \leq z^*(x, y) \leq v^{(m+1)}(x, y) \leq v^{(m)}(x, y)$$

hold true on $E^{*}_a$ for every $m \in \mathbb{N}$.

REMARK 1. Note that Theorem 4 only ensures the local existence of solutions of the semilinear problem (12),(13) on the set $E_a$, where $\bar{a} \in (0, a]$ depends on the Lipschitz functions of $G$ and on the constants $Q_0, Q_1, Q_2$. Each of the functions $u^{(m+1)}, v^{(m+1)}$ is a solution of a semilinear equation with different right-hand sides but since $u^{(m)}, v^{(m)}$ belong to the same space $C^{1,L}_a(Q)$, the Lipschitz functions for all these equations are the same and, consequently, both sequences are defined on the same set $E_{\bar{a}}$.

Now we prove that $\{u^{(m)}\}$ and $\{v^{(m)}\}$ are convergent to $z^*$ on $E^{*}_a$.

THEOREM 5. Suppose that Assumptions $H_5, H_8, H_9$ are satisfied and

1° the functions $u^{(0)}, v^{(0)}$ satisfy the inequality

$$|u^{(0)}(x, y) - v^{(0)}(x, y)| \leq A, \quad \text{on } E^{*}_a,$$

where $A = [4\bar{a}K_2 \exp\{K_1\bar{a}\}]^{-1}$;
2° the initial estimates

\[ |\alpha^{(m)}(x, y) - \beta^{(m)}(x, y)| \leq A \left[ 2^{2^m} \exp\{K_1 \tilde{a}\} \right]^{-1} \]

hold true on \( E_0^* \cup \partial_0E_0 \).

Then we have

\begin{equation}
|u^{(m)}(x, y) - v^{(m)}(x, y)| \leq \frac{2A}{2^{2m}}, \quad \text{on } E_0^*, \text{ for } m \in \mathbb{N}.
\end{equation}

Proof. We prove (15) by induction. It is immediately seen that (15) is satisfied for \( m = 0 \), suppose that it holds for some \( m \in \mathbb{N} \). Then for all \( (x, y) \in E_0 \) we have

\[
D_{x}v^{(m+1)}(x, y) - D_{x}u^{(m+1)}(x, y) \\
= \mathcal{H}(x, y, v^{(m+1)}(x, y), D_{x}v^{(m+1)}(x, y), u^{(m)}(x, y), v^{(m)}(x, y)) - \mathcal{G}(x, y, v^{(m+1)}(x, y), D_{y}u^{(m+1)}(x, y), u^{(m)}(x, y)) \\
= F(x, y, v^{(m)}(x, y)) + D_{w}F(x, y, u^{(m)}(x, y)) \circ (v^{(m+1)}(x, y) - v^{(m)}(x, y)) \\
+ \sum_{i=1}^{n} f_i(x, y)D_{y_i}v^{(m+1)}(x, y) - F(x, y, u^{(m)}(x, y)) \\
- D_{w}F(x, y, u^{(m)}(x, y)) \circ (u^{(m+1)}(x, y) - u^{(m)}(x, y)) - \sum_{i=1}^{n} f_i(x, y)D_{y_i}u^{(m+1)}(x, y) \\
= \left[ D_{w}F(x, y, u^{(m)}(x, y)) + \theta(v^{(m+1)}(x, y) - u^{(m+1)}(x, y)) - D_{w}F(x, y, u^{(m)}(x, y)) \right] \\
\circ (v^{(m)}(x, y) - u^{(m)}(x, y)) \\
+ D_{w}F(x, y, u^{(m)}(x, y)) \circ (v^{(m+1)}(x, y) - u^{(m+1)}(x, y)) \\
+ \sum_{i=1}^{n} f_i(x, y)(D_{y_i}v^{(m+1)}(x, y) - D_{y_i}u^{(m+1)}(x, y)).
\]

where \( \theta \in (0, 1) \). If for any \( m \in \mathbb{N} \) we write \( \tilde{w}^{(m)} = v^{(m)} - u^{(m)} \), then by Assumption \( H_8 \) and by the induction hypothesis we have

\[
\left| D_{x}(\tilde{w}^{(m+1)}(x, y) - \sum_{i=1}^{n} f_i(x, y)D_{y_i}\tilde{w}^{(m+1)}(x, y)) \right| \\
\leq K_2(\tilde{w}^{(m)}(x, y)\|_0^2 + K_1\|\tilde{w}^{(m+1)}(x, y)\|_0 \\
\leq K_1\|\tilde{w}^{(m)}(x, y)\|_0 + K_2\left( \frac{2A}{2^{2m}} \right)^{2}.
\]
The above estimate together with Lemma 2 yields

\[ |\tilde{w}^{(m+1)}(x, y)| \]
\[ \leq A\left[2^{2m+1} \exp\{K_1 \bar{a}\}\right]^{-1} \exp\{K_1 x\} + K_2 \left(\frac{2A}{2^m}\right)^2 \int_0^x \exp\{K_1(x - t)\} dt \]
\[ \leq \frac{A}{2^{2m+1}} + \frac{4A^2 K_2 \bar{a} \exp\{K_1 \bar{a}\}}{2^{2m+1}} = \frac{2A}{2^{2m+1}}. \]

Thus Theorem 5 follows by induction.

**Remark 2.** The inequalities (14),(15) imply the following error estimates of the differences between the terms of Chaplygin sequences and the solution \( z^* \)

\[ 0 \leq z^*(x, y) - u^{(m)}(x, y) \leq \frac{2A}{2^m}, \quad 0 \leq v^{(m)}(x, y) - z^*(x, y) \leq \frac{2A}{2^m}, \]

on \( E_\bar{a} \).

**References**


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