AN ESTIMATE OF THE GEOMETRIC DEGREE
OF AN EXTENSION OF SOME POLYNOMIAL
PROPER MAPPINGS

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Abstract. Let $V$, $W$ be algebraic subsets of $\mathbb{C}^n$, $\mathbb{C}^m$, respectively, with $n \leq m$. It is known that any proper regular mapping $f: V \to W$ can be extended to a proper regular mapping $F: \mathbb{C}^n \to \mathbb{C}^m$. In this paper we partially answer the question how big the geometric degree of an extension $F$ is compared with the geometric degree of $f$.

1. Introduction. Let $V$, $W$ be algebraic subsets of $\mathbb{C}^n$, $\mathbb{C}^m$, respectively. A regular mapping $f = (f_1, \ldots, f_n): V \to W$ induces a homomorphism $f^*: C[W] \to C[V]$ of the rings $C[W]$, $C[V]$ of regular functions on $W$, $V$ respectively, defined by $f^*(p) := p(f_1, \ldots, f_n)$. A regular mapping $f$ is called finite if $C[V]$ is an integral extension of $f^*(C[W])$. It is well known that a regular mapping is finite if and only if it is proper\(^1\) (see e.g. [No], Th.1).

We have the following

THEOREM 1 ([Kw]). Let $V$, $W$ be algebraic subsets of $\mathbb{C}^n$, $\mathbb{C}^m$, respectively, with $n \leq m$. If $f: V \to W$ is a proper regular mapping then there exists a proper regular mapping $F: \mathbb{C}^n \to \mathbb{C}^m$ such that $F|_V = f$.

If $f: V \to W$ is a proper regular mapping then there exists $d \in \mathbb{N}$ such that $\# f^{-1}(y) \leq d$ for all $y \in W$. A number of points in a generic fiber is called the geometric degree of the mapping $f$ and denoted by $\text{gdeg } f$. If $f: V \to W$, $g: W \to Z$ are proper regular mappings then $g \circ f$ is also a

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\(^1\)Inverses of compact sets are compact.
proper regular mapping and \( g \circ f \leq g \circ f \cdot g \circ g \) if \( f(V) = W \) then \( g \circ f = g \circ f \cdot g \circ g \).

For an irreducible algebraic set \( V \subset \mathbb{C}^n \) by \( \mathbb{C}(V) \) we will denote field of rational function on the set \( V \). We have the following

**Theorem 2** ([Mum] Th.3.17). Let \( V \subset \mathbb{C}^n \), \( W \subset \mathbb{C}^m \) be irreducible algebraic sets of the same dimension. Let \( f: V \to W \) be a dominating regular mapping. Then there exists a Zariski open subset \( U \) of \( W \) such that

\[
\# f^{-1}(y) = [\mathbb{C}(V): f^*(\mathbb{C}(W))] \quad \text{for } y \in U.
\]

Let now \( V \subset \mathbb{C}^n \) be an algebraic set of pure dimension \( k \). We know that there exists a linear change of coordinates in \( \mathbb{C}^n \) such that \( \Pi|_V \), where \( \Pi: \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k) \in \mathbb{C}^k \), is proper (see e.g. [Rud], it can also be deduced from Noether normalization lemma). Following Bezout’s theorem, the degree of \( V \), denoted by \( \text{Deg} V \), can be characterized as the smallest number \( d \in \mathbb{N} \) such that for any linear change of coordinates in \( \mathbb{C}^n \), for which \( \Pi|_V \) is proper, \( g \circ \text{Deg}(\Pi|_V) \leq d \).

Let us fix a proper regular mapping \( f: V \to W \). By theorem 1 we know that there exists a proper regular extension \( F: \mathbb{C}^n \to \mathbb{C}^m \) of \( f \). We try to estimate from above the number

\[
\min \{ \text{gdeg} F: F \text{ is a proper regular extension of } f \}.
\]

To illustrate this problem consider the following

**Example.** For any number \( k \in \mathbb{N} \setminus \{0\} \) let \( A_k := \{0, 1, \ldots, k\} \subset \mathbb{C} \) and \( f_k := \{ \mathbb{C} \ni z \mapsto z^k \in \mathbb{C} \}|_{A_k} \). The mapping \( f_k: A_k \to \mathbb{C} \) is proper regular and \( \text{gdeg} f_k = 1 \) for all \( k \in \mathbb{N} \setminus \{0\} \). The simple check proves that for all \( k \in \mathbb{N} \setminus \{0\} \):

\[
\min \{ \text{gdeg} F: F \text{ is a proper regular extension of } f_k \} = k.
\]

Let \( V \subset \mathbb{C}^n \) and \( f: V \to \mathbb{C}^n \) be a proper regular mapping and \( \text{gdeg} f = 1 \). The existence of a regular extension of \( f \) with the same geometric degree, means in this situation that there exists a regular automorphism \( F \) of \( \mathbb{C}^n \) such that \( f \) is the restriction of this automorphism to the set \( V \). In [Jel] it was shown that if \( \dim V \) is sufficiently small relatively to \( n \), and if \( V \) has “nice” singularities then any regular (also proper) embedding of \( V \) into \( \mathbb{C}^n \) is a restriction of some automorphism to the set \( V \).

2. **Decomposition of proper regular mappings.** Let now \( V \) be an irreducible subset of \( \mathbb{C}^n \) and \( f = (f_1, \ldots, f_m): V \to \mathbb{C}^m \) be a proper regular mapping. Since \( f \) is proper, the set \( W := f(V) \) is algebraic in \( \mathbb{C}^m \) and
$f^* : \mathbf{C}[W] \to \mathbf{C}[V]$ is a ring monomorphism. Since also $\mathbf{C}[V]$ is an integral extension of $f^*(\mathbf{C}[W]) = \mathbf{C}[f_1, \ldots, f_m]$ and $\mathbf{C}[f_1, \ldots, f_m][x_1, \ldots, x_n] = \mathbf{C}[V]$, where $x_1, \ldots, x_n$ are coordinate restrictions to $V$, there exist $a_1, \ldots, a_n \in \mathbf{C}$ such that $l = a_1 x_1 + \ldots + a_n x_n$, is a primitive element of the extension $\mathbf{C}[V]$ over the ring $\mathbf{C}[f_1, \ldots, f_m]$ (It follows from Th A.8 in [Loj]).

**Theorem 3.** Let $V \subset \mathbf{C}^n$ be an irreducible algebraic set and $f : V \to \mathbf{C}^m$ a proper regular mapping and $l$ as above. Then

(1) $\psi : V \ni x \mapsto (f(x), l(x)) \in \psi(V) \subset \mathbf{C}^{m+1}$ is a proper regular mapping and $\operatorname{gdeg} \psi = 1$,

(2) $p : \psi(V) \ni (x_1, \ldots, x_{m+1}) \mapsto (x_1, \ldots, x_m) \in \mathbf{C}^m$ is proper and $\operatorname{gdeg} p = \operatorname{gdeg} f$.

**Proof.** Let $W := f(V)$. We have the following commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\psi} & \psi(V) \\
\downarrow f & & \downarrow p \\
W & & 
\end{array}
\]

To prove 1, observe that

\[
(1) \quad \psi^{-1}(K) = f^{-1}(p(K)) \cap \psi^{-1}(K)
\]

for any compact set $K \subset \psi(V)$. Since the right hand side of (1) is compact (as an intersection of a compact and a closed sets), $\psi^{-1}(K)$ is compact, and so $\psi$ is proper. Since $l$ is a primitive element of the extension $\mathbf{C}(V) \supset f^*(\mathbf{C}(W))$, $\mathbf{C}(V) = f^*(\mathbf{C}(W))(l) = \mathbf{C}(f_1, \ldots, f_m, l) = \psi^*(\mathbf{C}(\psi(V)))$. By Theorem 2 we conclude that $\operatorname{gdeg} \psi = 1$.

To prove 2, observe that

\[
p^{-1}(Z) = \psi(f^{-1}(Z))
\]

for any compact $Z \subset \mathbf{C}^m$ so the set $p^{-1}(Z)$ is compact and then $p$ is proper. Since $\psi^* : \mathbf{C}(\psi(V)) \to \mathbf{C}(V)$ is an isomorphism,

\[
[C(V) : f^*(C(W))] = [\psi^*(\mathbf{C}(\psi(V))) : \psi^*(p^*(\mathbf{C}(W)))]
\]

and so, by Theorem 2, $\operatorname{gdeg} f = \operatorname{gdeg} p$. \qed
3. Proper regular extensions in a few special cases.

**Example.** Let $W \in \mathbb{C}[X_1, \ldots, X_n]$ be a polynomial such that

$$p|_{W^{-1}(0)} : W^{-1}(0) \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$$

is proper. The mapping $\Pi := (p, W) : \mathbb{C}^n \to \mathbb{C}^n$ is a proper regular extension of $p|_{W^{-1}(0)}$ and $\deg \Pi \leq \deg_{x_n} W$.

**Theorem 4.** Let $V \subset \mathbb{C}^n$ be a hypersurface and $p : V \to \mathbb{C}^{n-1}$ be the natural projection. If $p$ is proper then there exists a proper regular extension $\Pi : \mathbb{C}^n \to \mathbb{C}^n$ of $p$ such that

$$\deg \Pi = \deg p.$$

**Proof.** Let $W \in \mathbb{C}[X_1, \ldots, X_n]$ be a generator of the ideal of all polynomials vanishing on $V$. It is easy to check that $D_{x_n}(W) \neq 0$ where $D_{x_n}(W)$ is a discriminant of the polynomial $W$ with respect to $x_n$. Since $D_{x_n}(W) \neq 0$, $\deg_{x_n} W = \deg p$ and by the example we have that $\Pi : \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, W(x_1, \ldots, x_n)) \in \mathbb{C}^n$ is good.

**Theorem 5.** Let $V \subset \mathbb{C}^n$ be an irreducible hypersurface. Let $f : V \to \mathbb{C}^{n-1}$ be a proper regular mapping and let $l \in \mathbb{C}[V]$ be a primitive element of the extension $\mathbb{C}(V) \supset \mathbb{C}(f_1, \ldots, f_n)$ of the form $l = a_1 x_1 + \ldots + a_n x_n$, where $a_1, \ldots, a_n \in \mathbb{C}$.

If $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ is a proper regular extension of the mapping $\psi$ given by

$$\psi : V \ni x \mapsto (f(x), l(x)) \in \psi(V) \subset \mathbb{C}^{n-1} \times \mathbb{C}$$

then there exists a proper regular extension $F : \mathbb{C}^n \to \mathbb{C}^n$ of $f$ such that

$$\deg F \leq \deg \Psi \cdot \deg f.$$  

**Proof.** Let $p : \psi(V) \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$ be the natural projection. By Theorem 3 we know that $\psi$ is a proper regular mapping and then $\psi(V)$ is a hypersurface. We know also that $p$ is proper, $\deg p = \deg f$ and by Theorem 4 there exists a proper regular extension $\Pi : \mathbb{C}^n \to \mathbb{C}^n$ of $p$ such that $\deg \Pi = \deg p = \deg f$. The mapping $F := \Pi \circ \Psi$ is a proper regular extension of $f$ and (2) holds.

Now we consider extensions from affine subspaces.
THEOREM 6. Let \( V \) be an affine subspace of \( \mathbb{C}^n \) of dimension \( k < n \) and \( f = (f_1, \ldots, f_n) : V \to \mathbb{C}^n \) be a proper regular mapping. Then there exists a proper regular extension \( F : \mathbb{C}^n \to \mathbb{C}^n \) of \( f \) such that
\[
\text{gdeg} F \leq \text{gdeg} f \cdot \text{Deg}(f(V)).
\]

PROOF. Without a loss of generality we can assume that \( V \) is given by \( x_{k+1} = \ldots = x_n = 0 \) and \( p : f(V) \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k) \in \mathbb{C}^k \) is proper. We have: \( \text{gdeg} p \leq \text{Deg}(f(V)) \) and \( p \circ f = (f_1, \ldots, f_k) \). We also have: \( \text{gdeg}(p \circ f) \leq \text{gdeg} p \cdot \text{gdeg} f \leq \text{Deg}(f(V)) \cdot \text{gdeg} f \). The mapping
\[
F := (f_1, \ldots, f_k, f_{k+1} + x_{k+1}, \ldots, f_n + x_n)
\]
is a proper extension of \( f \) to \( \mathbb{C}^n \) and the condition (3) holds. \( \square \)

REMARK 7. If there exists a natural proper projection \( q : f(V) \to \mathbb{C}^k \) such that \( \text{gdeg} q < \text{Deg}(f(V)) \) then there exists a proper regular extension \( F : \mathbb{C}^n \to \mathbb{C}^n \) of \( f \) such that
\[
\text{gdeg} F < \text{gdeg} f \cdot \text{Deg}(f(V)).
\]

In the following example, we will show that the geometric degree of the extension \( F \) given by (4) depends on the choice of a projection \( p : f(V) \to \mathbb{C}^k \).

EXAMPLE. The mapping \( f : \mathbb{C} \ni t \mapsto (t^2, t^3) \in \mathbb{C}^2 \) is the bijective proper mapping onto the set \( V \) given by \( y^2 = x^3 \). We have: \( \text{Deg} V = 3 \). If we extend \( f \) to \( \mathbb{C}^2 \) using (4) with respect to \( p_y : V \ni (x, y) \mapsto y \in \mathbb{C} \) we get
\[
\text{gdeg} F_y = 3 = \text{gdeg} f \cdot \text{Deg} V
\]
and with respect to \( p_x : V \ni (x, y) \mapsto x \in \mathbb{C} \) we get
\[
\text{gdeg} F_x = 2 < \text{deg} f \cdot \text{Deg} V.
\]

References


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