AN INVARIANCE THEOREM 
FOR STOCHASTIC FUNCTIONAL 
DIFFERENTIAL EQUATIONS 

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Abstract. The aim of the present paper is to extend an invariance theorem 
to the stochastic functional differential equations. The necessary and sufficient 
condition for the invariance property is given in the terms of contingent cones. 

1. Introduction. In this paper we examine an invariance theorem for 
the stochastic functional differential equations. More precisely, we give the 
necessary and sufficient conditions for the invariance of the solutions to these 
equations using the contingent sets introduced by Aubin and Da Prato ([2]). 
In our proof we follow the idea of Milian ([11]) based there on the Stroock and 
Varadhan support theorem ([13]). But in the present paper we use the support 
theorem for the stochastic functional differential equations of Dawidowicz and 
Twardowska ([4]). We also use the formula for the transition from the Itô 
to the Stratonovich equations with integrands of delayed argument proved by 
Dawidowicz and Twardowska in [3]. 

The invariance theorem for the deterministic differential equations was con-
sidered by Nagumo ([1]). The theorem of such a type for the deterministic 
functional differential equations was proved by Jachimiak in [7]. 

Some invariance theorems for stochastic differential equations in finite-
dimensional case were given by Milian, e.g., in [9]-[11]. More precisely, in [11] 
Milian gives the necessary and sufficient conditions for the invariance property 
of a closed subset of $\mathbb{R}^m$, expressed in terms of properly defined contingent 
cones. The tools used in this paper are the Stroock and Varadhan theorem 
([13]) and the Nagumo theorem ([1]). Assuming that $K$ is a manifold of $\mathbb{R}^m$, 

the criterion is expressed in local coordinates. Criteria for the viability and invariance given in [2] are generalized by Milian in [9] to arbitrary subsets which can be also time-dependent and random. However, the conditions expressed by the contingent sets are general but not easy to check so in [10] some checkable conditions are given for the sets \( K \) of \( \mathbb{R}^m \) which are polyhedrons.

2. Definitions and auxiliary results. Let \( t \in [0, T] \) and let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a complete probability space with an increasing family \( \mathcal{F}_t = (\mathcal{F}_t)_{t \in [0, T]} \) of sub-\( \sigma \)-algebras of the \( \sigma \)-algebra \( \mathcal{F} \). We put \( J = [-r, 0] \) and we introduce the spaces \( C_\sigma = C(J, \mathbb{R}^d), C_1 = C([-r, T], \mathbb{R}^d) \) and \( C_2^0 = C([0, T], \mathbb{R}^m) = \bar{\Omega} \) of continuous functions. The spaces \( C_\sigma, C_1 \) and \( C_2^0 \) are endowed with the usual norms of uniform convergence. Here \( d \) is the dimension of the state space and \( m \) is the dimension of the Wiener process, in the space \( C_2^0 \) all functions are equal to zero at zero.

Let \( \mathcal{B}(C_2^0) \) denote the Borel \( \sigma \)-algebra of the space \( C_2^0 \). It is obvious that it is identical to the \( \sigma \)-algebra generated by the family of all Borel cylinder sets in \( C_2^0 \) (compare Proposition 4.1, [5], Chapter I). So we construct the Wiener space \((C_2^0, \mathcal{B}(C_2^0), P^w)\), where \( P^w \) is the Wiener measure ([5], Chapter I). The coordinate process \( B(t, w) = w(t), w \in C_2^0 \), is the \( m \)-dimensional Wiener process.

The smallest Borel \( \sigma \)-algebra that contains \( \mathcal{B}_1 \), \( \mathcal{B}_2, \ldots \) is denoted by \( \mathcal{B}_1 \vee \mathcal{B}_2 \vee \ldots \) or \( \vee_{\alpha} \mathcal{B}_\alpha \); \( \mathcal{B}_{u,v}(X) \) denotes the smallest Borel \( \sigma \)-algebra for which a given stochastic process \( X(t) \) is measurable for every \( t \in [u, v] \) and \( \mathcal{B}_{u,v}(dB) \) denotes the smallest Borel algebra for which \( B(s) - B(t) \) is measurable for every \( (t, s) \) with \( u \leq t \leq s \leq v \).

Let \( B^n(t, w) = w^n(t) \) be the following piecewise linear \( \mathcal{F}_t \)-adapted (because of the shift of time in \((*)\)) approximation of \( B(t, w) = w(t) \):

\[
B^{n,p}(t, w) = w^p \left( \frac{k}{2^n} \right) + 2^n (t - \frac{k}{2^n}) (w^p \left( \frac{k+1}{2^n} \right) - w^p \left( \frac{k}{2^n} \right))
\]

for each \( p = 1, \ldots, m \) and

\[
(*) \quad \frac{(k+1)T}{2^n} \leq t < \frac{(k+2)T}{2^n}
\]

for \( k = 0, 1, \ldots, 2^n - 1 \).

For the stochastic process \( X(t, w) \) and for a fixed \( t \in [0, T] \) we define

\[
X_t(\theta, w) = X(t + \theta, w), \theta \in J;
\]

therefore \( X_t(\cdot, w) \) denotes the segment of the trajectory \( X(\cdot, w) \) on \([t-r, t]\).
Now we consider $\tilde{\Omega} = C^0_2$. Let $X$ be a continuous stochastic process $X(t, w) : [-r, T] \times \tilde{\Omega} \to \mathbb{R}^d$, that is, $X : \tilde{\Omega} \to C_1$.

We take some fixed initial constant stochastic processes $X^i(0 + \theta, w) = X^i_0(w) = X^i_n(\theta) = Y^i_0(\theta)$ for $\theta \in J, i = 1, \ldots, d$. We also consider operators $b : C_- \to \mathbb{R}^d, \sigma : C_- \to L(\mathbb{R}^m, \mathbb{R}^d)$ (where $L(\mathbb{R}^m, \mathbb{R}^d)$ is the Banach space of linear functions from $\mathbb{R}^m$ to $\mathbb{R}^d$ with uniform operator norm $| \cdot |_L$, that is, for $A : \mathbb{R}^m \to \mathbb{R}^d$, $x \in \mathbb{R}^m$ and $(Ax)_i = \sum_{j=1}^m a_{ij} x_j$ we put $| A |_L = \sup_{|x|_{\mathbb{R}^m} = 1} |Ax|_{\mathbb{R}^d}$).

We consider the following stochastic functional differential equation of Itô type

\[(2.2) \quad X^i(t, w) = X^i_0(w) + \int_0^t b^i(X_s(\cdot, w))ds + \sum_{p=1}^m \int_0^t \sigma^i_p(X_s(\cdot, w))dw^p(s)\]

for $i = 1, \ldots, d$.

We recall the Stratonovich form of equation (2.2) (see [3])

\[(2.3) X^i(t, w) = X^i_0(w) + \int_0^t b^i(X_s(\cdot, w))ds + \sum_{p=1}^m \int_0^t \sigma^i_p(X_s(\cdot, w))dw^p(s)\]

\[+ \frac{1}{2} \sum_{p=1}^m \sum_{j=1}^d \int_0^t \tilde{D}_j \sigma^i_p(X_s(\cdot, w))\sigma^j_p(X_s(\cdot, w))ds\]

and the Stratonovich integral

\[\sum_{p=1}^m \int_0^t \sigma^i_p(X_s(\cdot, w)) \circ dw^p(s) = \sum_{p=1}^m \int_0^t \sigma^i_p(X_s(\cdot, w))dw^p(s)\]

\[+ \frac{1}{2} \sum_{p=1}^m \sum_{j=1}^d \int_0^t \tilde{D}_j \sigma^i_p(X_s(\cdot, w))\sigma^j_p(X_s(\cdot, w))ds.\]

Further, $D\sigma^i_p$ is the Fréchet derivative from $C_-$ to $L(C_-, \mathbb{R})$. From the Riesz Theorem (see Rudin [12]) it follows that there exists a family of measures $\mu = \mu^i_{g^j_p}$ with bounded variation and such that

\[D\sigma^i_p(g)(\Phi) = \sum_{j=1}^d \int_{-r}^0 \Phi_j(v) \mu^i_{g^j_p}(dv)\]

for any $g, \Phi \in C_-$. Then $D\sigma^i_p(g)(\Phi)$ is a directional derivative at the point $g$ in the direction $\Phi$. The measure $\mu$ has the following decomposition:

\[\mu(A) = \mu(A \cap [-r, 0]) + \mu(A \cap \{0\}) = \tilde{\mu}(A) + \mu(\{0\})\delta_0(A),\]
where $\delta_0$ is the Dirac measure, $A \in \mathcal{B}([-r,0))$. We denote the value $\mu^{ijp}_{y}(\{0\})$ by $\widetilde{D}_j\sigma^{jp}(g)$, that is, $\widetilde{D}_j\sigma^{jp}(X_s(\cdot,w)) = \mu^{ijp}_{s,w,x}(\{0\})$.

Let us introduce the following conditions:

$(\tilde{A}1)$ The initial stochastic process $X_0$ is $\mathcal{F}_0$-measurable and $P(|X_0(w)| = \infty) = 1$, where $|X_0(w)| = \sum_{i=1}^{d} |X^i_0(w)|$, $\mathcal{B}_{-r,0}(X_0)$ is independent of $\mathcal{B}_{0,T}(dB)$.

$(\tilde{A}2)$ For every $\varphi, \psi \in C_-$ the following Lipschitz condition is satisfied:

$$|b(\varphi) - b(\psi)|^2 + |\sigma(\varphi) - \sigma(\psi)|^2_L \leq L^1 \int_{-r}^{0} |\varphi(\theta) - \psi(\theta)|^2 dK(\theta) + L^2 |\varphi(0) - \psi(0)|^2,$$

where $K(\theta)$ is a certain bounded measure on $J$, and $L^1, L^2$ are some constants.

$(\tilde{A}3)$ For every $\varphi, \psi \in C_-$ the following growth condition is satisfied:

$$|b(\varphi)|^2 + |\sigma(\varphi)|^2_L \leq L^1 \int_{-r}^{0} (1 + \varphi^2(\theta))dK(\theta) + L^2(1 + \varphi^2(0)),$$

where $\varphi^2(0) = \sum_{i=1}^{d} \varphi_i^2(0)$.

$(\tilde{A}4)$ $b^i, \sigma^{ip} \in C^1_b(C_\cdot)$, for every $i = 1, \ldots, d$, $p = 1, \ldots, m$, where $C^1_b$ denotes the space of bounded mappings with continuous bounded first derivative, and the first derivatives of $\sigma^{ip}$ satisfy the Lipschitz condition.

Notice that conditions $(\tilde{A}1)$–$(\tilde{A}3)$ ensure the existence and uniqueness of solution to equation (2.2) (compare [6], [8], [15]).

Let $P_X$ be the probability law of the solution $X = \{X(t)\}, t \in [0,T]$, to equation (2.2). Let $\mathcal{F}$ be the space of functions $h : [0,T] \to \mathbb{R}^m$ that are absolutely continuous and whose derivatives $\dot{h}$ belong to $L^2([0,T], \mathbb{R}^m)$. Let

$$\mathcal{S}_\mathcal{F} = \{ h \in \mathcal{F} : h(0) = 0 \}.$$

For given $h \in \mathcal{S}_\mathcal{F}$ and a constant initial condition $X_0(w) = X_0 \in \mathbb{R}^d$ we consider the following deterministic equation

$$\xi^i(t) = X^i_0 + \int_0^t b^i(\xi(s))ds + \sum_{p=1}^{m} \int_0^t \sigma^{ip}(\xi(s))\dot{h}^p(s)ds$$

$$- \frac{1}{2} \sum_{p=1}^{m} \sum_{j=1}^{d} \int_0^t \tilde{D}_j\sigma^{jp}(\xi(s))\sigma^{jp}(\xi(s))ds$$

for every $i = 1, \ldots, d$. 

(2.4)
Let

\begin{align}
S_1 &= \text{supp } P_x \text{ in } G = C_1, \\
S_2 &= \overline{\{\xi = \xi(x,h) : h \in S_3}\}} \text{ (closure in } G) .
\end{align}

We proved in [4] the following

**Theorem 2.1.** Let \( \sigma \) and \( b \) be functions satisfying conditions \((\tilde{A}1)-(\tilde{A}4)\) and let \( X(t) \) be the solution to equation (2.2) with a constant initial condition. Let \( S_1 \) and \( S_2 \) be given by (2.5) and (2.6), respectively. Then \( S_1 = S_2 \).

For a subset \( K \) of \( \mathbb{R}^d \) and for a subset \( \tilde{K} \) of \( \mathbb{C}_- \) we introduce the following notions.

**Definition 2.1.** Let \( \tilde{K} \) be a subset of \( \mathbb{C}_- \) and \( I \subset \mathbb{R} \) an interval. A stochastic process \( X(t), t \in I \), is viable in \( \tilde{K} \) on \( I \) if and only if

\[ P\{X_t(\cdot) \in \tilde{K}, \ t \in I\} = 1. \]

**Definition 2.2.** A set \( \tilde{K} \subset \mathbb{C}_- \) is invariant with respect to equation (2.2) if every solution \( X(\cdot) \) to (2.2), starting at any point \( x \in \tilde{K} \), is viable in \( \tilde{K} \) on the interval \([0, \infty)\).

Thus the viability problem consists in characterizing a fixed subset and an equation such that the equation has viable trajectories in the subset for any initial state from the subset. The problem to characterize a subset and an equation in order for each solution to the equation starting from the subset to be viable in the subset, is the so-called invariance problem.

**Definition 2.3.** Let \( K \) be a nonempty subset of \( \mathbb{R}^d \) and \( x \in K \). We define the contingent cone to \( K \) at \( x \) by:

\begin{align}
T_K(x) &= \{v \in \mathbb{R}^d : \liminf_{h \downarrow 0} \frac{d_K(x + hv)}{h} = 0\},
\end{align}

where \( d_K(x) \) is the distance defined by

\[ d_K(x) = \inf\{|x - y|, y \in K\}. \]

**Definition 2.4.** Let \( K \) be a nonempty subset of \( \mathbb{R}^d \) and \( x \in K \). The \( m \)-contingent cone to \( K \) at \( x \in K \) is the set

\begin{align}
T_{K,m} &= \{\beta, v \in \mathbb{R}^d \times \mathbb{R}^{d \times m} : \forall \lambda_1, \ldots, \lambda_m \in \mathbb{R}, \ \beta + \sum_{j=1}^m \lambda_j v^j \in T_K(x)\},
\end{align}

where \( v = [v_{ij}] \) is the matrix and \( v^j \) is the \( j \)-th column of \( v \), \( j = 1, \ldots, m \).
The above definition contains of course the case of deterministic functions $X(\cdot)$.

Consider the following deterministic funtional differential equation

\begin{equation}
 x'(t) = f(x_t), \ t \in [0, \alpha), \ \alpha > 0, \ x_0 = \varphi \in \mathcal{C}_-.
\end{equation}

Further, we have

**Definition 2.5.** Let $K$ be a given subset of $\mathbb{R}^d$. The fully constrained set is given by the formula

\[ F(K) = \{ \varphi \in \mathcal{C}_- : \varphi(s) \in K, \ s \in [-r, 0] \}. \]

The proposition below gives a necessary and sufficient condition for the viability of fully constrained sets.

**Proposition 2.2.** ([7]). Let $K$ be a nonempty closed subset of $\mathbb{R}^d$ and let $f : \mathcal{C}_- \to \mathbb{R}^d$ be a continuous function. If for any $\varphi \in F(K)$ the equation

\begin{equation}
 x'(t) = f(x_t), \ t \geq 0, \ x_0 = \varphi \in F(K)
\end{equation}

has a solution viable in $F(K)$, then the following condition

\begin{equation}
 \lim_{h \downarrow 0} \frac{d_K(x(0) + hf(x))}{h} = 0, \ x \in F(K)
\end{equation}

holds.

We quote from [7]

**Theorem 2.3.** Let $K$ be a nonempty, closed, convex subset of $\mathbb{R}^d$, and let $f : \mathcal{C}_- \to \mathbb{R}^d$ be a Lipschitz mapping. Then equation (2.10) has a viable solution in $F(K)$ for any $\varphi \in F(K)$ if and only if condition (2.11) holds.

### 3. Invariance theorem.

We shall state our main result.

**Theorem 3.1.** Let assumptions (\tilde{A}1)-(\tilde{A}4) be satisfied. Let $K$ be a closed convex subset of $\mathbb{R}^d$. Then $F(K)$ is the invariant set with respect to equation (2.2) with a constant initial condition if and only if

\begin{equation}
 (b(x) + s(x), \sigma(x)) \in T_{K,m}(x) \text{ for every } x \in F(K),
\end{equation}

where $s = (s_1, \ldots, s_d)$ is defined by

\begin{equation}
 s_i = -\frac{1}{2} \sum_{p=1}^{m} \sum_{j=1}^{d} \tilde{D}_j \sigma^p(x) \sigma^{jp}(x), \ i = 1, \ldots, d.
\end{equation}
PROOF. We write (2.2) in the Stratonovich form:

\[(3.3) \quad X^i(t, w) = X^i_0 + \int_0^t \alpha^i(X^i_s(\cdot, w))ds + \sum_{p=1}^m \int_0^t \sigma^{ip}(X^i_s(\cdot, w)) \circ dw^p(s)\]

for \(i = 1, \ldots, d\), where \(\alpha(x) = b(x) + s(x)\). So we observe that condition (3.1) is equivalent to the condition

\[(3.4) \quad (\alpha(x), \sigma(x)) \in T_{K,m}(x) \text{ for every } x \in F(K)\]

as well as that the set \(F(K)\) is invariant with respect to (2.2) if and only if it is invariant with respect to (3.3).

Now by the support theorem (i.e. by Theorem 2.1) for the functional stochastic differential equations we deduce that the set \(F(K)\) is invariant with respect to equation (3.3) if and only if the following condition is satisfied:

\[(3.5) \quad \text{for each } x \in F(K) \text{ and each function } h \in \mathcal{S}_0\]
\[
\text{the solution } \xi(x, h) \text{ to (2.4) is viable in } F(K) \text{ on } [0, \infty).
\]

To finish the proof it is sufficient to show that conditions (3.4) and (3.5) are equivalent. First assume that (3.5) holds. Let \(x \in F(K)\) and \(\lambda_1, \ldots, \lambda_m \in \mathbb{R}\) be arbitrary positive numbers. For a piecewise smooth function
\(h(s) = (\lambda_1, \ldots, \lambda_m)\) and \(t \geq 0\) we have

\[
\xi^i(t) = X^i_0 + t\{ \frac{1}{t} \int_0^t \alpha^i(\xi^i_s(\cdot))ds + \frac{1}{t} \sum_{p=1}^m \int_0^t (\sigma^{ip}(\xi^i_s(\cdot))) \}
-
\sigma^{ip}(X^i_0)\dot{h}^p(s)ds + \frac{1}{t} \sum_{p=1}^m \int_0^t \sigma^{ip}(X^i_0)\dot{h}^p(s)ds \} \in K
\]

for every \(i = 1, \ldots, d\). But we have, by the simple properties of integrals, that

\[
\frac{1}{t} \int_0^t \alpha^i(\xi^i_s(\cdot))ds \rightarrow \alpha^i(x) \text{ as } t \rightarrow 0,
\]

\[
\frac{1}{t} \sum_{p=1}^m \int_0^t (\sigma^{ip}(\xi^i_s(\cdot)) - \sigma^{ip}(X^i_0))\dot{h}^p(s)ds \rightarrow 0 \text{ as } t \rightarrow 0,
\]

\[
\frac{1}{t} \sum_{p=1}^m \int_0^t \sigma^{ip}(X^i_0)\dot{h}^p(s)ds = \sum_{j=1}^m \lambda_j \sigma^j(x).
\]
Since \( x \in F(K) \) we conclude from Proposition 2 in [1], p.177 that

\[
\alpha(x) + \sum_{j=1}^{m} \lambda_j \sigma_j(x) \in T_K(x).
\]

So we have obtained (3.4).

Now we assume that (3.4) holds and \( x \in F(K) \) is fixed. It is enough to show (3.5) for each piecewise constant function \( \bar{h}(s) \). Since (3.6) holds for each \( x \in F(K) \), that is,

\[
\alpha(x) + \sum_{j=1}^{m} \lambda_j \sigma_j(x) = v(\in \mathbb{R}^d) \in T_K(x), \ x \in F(K)
\]

so from (2.7) we get

\[
\lim_{h \downarrow 0} \frac{d_K(x(0) + hv)}{h} = 0, \ x \in F(K).
\]

Then from Theorem 2.3 we deduce that the deterministic solution \( \xi \) to (2.4) is viable in \( F(K) \), so (3.5) is satisfied, which completes the proof. \( \square \)

**Remark 3.1.** From the proof of Theorem 3.1 we can observe that the given closed subset \( \bar{K} \) of \( C_- \) is invariant with respect to equation (3.3) if and only if condition (3.4) is satisfied.

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**References**


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