CAPACITARY ESTIMATES
FOR THE GROUND-STATE SHIFT

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Abstract. Let \( H \geq 0 \) be a selfadjoint operator in \( L^2(X, m) \) corresponding to a regular and irreducible Dirichlet form such that the spectrum of \( H \) is purely discrete. If one imposes Dirichlet boundary conditions on a compact subset of \( X \), the first eigenvalue is shifted to the right. It is shown that the magnitude of this shift can by estimated by the capacity associated to the so-called ground-state transformation of \( H \).

0. Introduction. Let \((\mathcal{E}, \mathcal{F})\) be a regular, irreducible Dirichlet form on \( L^2(X, m) \) such that the associated non-negative definite selfadjoint operator has purely discrete spectrum and the semigroup is ultracontractive. If one imposes Dirichlet boundary conditions on a compact subset \( K \) of \( X \) the eigenvalues are of course shifted to the right. I. McGillivray has shown (see [4]) that the magnitude of this shift can be estimated by the capacity of \( K \), more precisely

\[
\lambda^\gamma - \lambda \leq c \text{cap}(K),
\]

where \( Y := X \setminus K \), \( \lambda \) is the unperturbed and \( \lambda^\gamma \) the perturbed ground-state energy. This result is only valid if \( \text{cap}(K) \) is sufficiently small. If \( K \) comes close to the boundary of \( X \) one must expect that the capacity increases, so the result in [4] can not be applied. Following a suggestion of E.B. Davies, we aim to surmount this shortcoming by introducing the so-called ground-state transformation. We define the unitary operator

\[
U : L^2(X, m) \to L^2(X, \phi^2m)
\]

\[
f \mapsto \phi^{-1}f,
\]
where $\phi$ is the ground-state, i.e. the eigenfunction corresponding to the lowest eigenvalue which is, due to irreducibility, always simple. Furthermore $\phi(x) > 0$ almost everywhere on $X$, see [1], Prop. 1.4.3. In this way we obtain a new operator $H' = UHU^{-1}$ having the same spectral properties as $H$; however the associated form $\mathcal{E}'$ induces a new capacity $\text{cap}'(\cdot)$ which is smaller on sets where $\phi$ is less than one. Since in many applications one has $\phi(x) \to 0$ as $x$ tends to the boundary of $X$, we have

$$\text{cap}'(K) \leq \text{cap}(K),$$

if $K$ is close to $\partial X$. We show that one can obtain (1) for the transformed capacity $\text{cap}'(\cdot)$: Dealing with $H'$ instead of $H$ has two advantages. Firstly the estimate (2) becomes better if $\text{cap}(\cdot)$ is replaced by $\text{cap}'(\cdot)$ and $K$ is close to the boundary. Secondly the ultraccontractivity condition required in [4] can be omitted since the transformed ground-state is identically equal to one and the above ultraccontractivity requirement was only needed to guarantee boundedness of the ground-state.

1. The result of McGillivray. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(X, m)$ and $H \geq 0$ the associated selfadjoint operator. The $\mathcal{E}$-capacity (simply denoted by $\text{cap}(\cdot)$) of an open subset $U$ of $X$ is defined by

$$\text{cap}(U) := \inf\{\mathcal{E}_1(u, u) : u \in \mathcal{F}, u \geq 1 \text{ m-a.e. on } U\},$$

where $\mathcal{E}_1(\cdot, \cdot)$ is the scalar product on $\mathcal{F}$ defined by

$$\mathcal{E}_1(u, u) := \mathcal{E}(u, u) + \|u\|^2, \quad (u \in \mathcal{F})$$

under which $(\mathcal{F}, \mathcal{E}_1(\cdot, \cdot))$ becomes a Hilbert space. If there is no such $u$ in (3) we define $\text{cap}(U)$ equal to $+\infty$. The capacity can be extented from the open subsets of $X$ to arbitrary $A \subset X$ by

$$\text{cap}(A) := \inf\{\text{cap}(U), U \supset A \text{ open}\}.$$ 

We now describe the assumptions on $\mathcal{E}$ as formulated in [4].

(a) Regularity of $\mathcal{E}$. This means by definition the density of $C_0(X) \cap \mathcal{F}$ in $(C_0(X), \|\cdot\|_\infty)$ as well as in $(\mathcal{F}, \mathcal{E}_1(\cdot, \cdot))$. Here $C_0(X)$ denotes the set of continuous functions on $X$ with compact support. Regularity has the consequence that every $u \in \mathcal{F}$ admits a quasicontinuous $m$-version $\tilde{u}$, two of which coincide quasi-everywhere (q.e.), see [2].

(b) Discreteness of the spectrum of $H$. It is required that each $\lambda \in \sigma(H)$ is an eigenvalue of finite multiplicity and that these eigenvalues do not accumulate.
(c) Irreducibility of the semigroup $e^{-tH}$. Let $A$ be a subset of $X$ with the property

$$
\chi_A e^{-tH} f = e^{-tH} (\chi_A f) \quad \text{for all } f \in L^2(X, m).
$$

Irreducibility means that (4) implies $m(A) = 0$ or $m(X \setminus A) = 0$ and has important consequences for the bottom of the spectrum of $H$, namely:

(i) The bottom eigenvalue $\lambda$ is simple.

(ii) The (normalized) eigenfunction $\phi$ corresponding to the ground-state energy $\lambda$ may be chosen such that $\phi(x) > 0$ m.a.e. on $X$, (see [1], Proposition 1.4.3). In many applications, e.g. if $H = -\Delta$ in $L^2(\Omega)$ with Dirichlet boundary conditions, $\Omega \subset \mathbb{R}^d$ a region with $C^\infty$-boundary, one also has

(iii) $\phi(x) \to 0$ as $x$ tends to the boundary of $X$, see [1], Section 4.

(d) Ultracontractivity of $e^{-tH}$, i.e. for each $t > 0$ there is a finite constant $c_t$ such that

$$
\|e^{-tH} f\|_\infty \leq c_t \|f\|_2
$$

for every $f \in L^2(X, m)$. Using inequality (5) we obtain the following estimate for the ground-state $\phi$:

$$
\|\phi\|_\infty = e^{\lambda t} \|e^{-\lambda t} \phi\|_\infty \\
= e^{\lambda t} \|e^{-tH} \phi\|_\infty \\
\leq c_t e^{\lambda t} \|\phi\|_2 \\
= c_t e^{\lambda t}.
$$

the ultracontractivity condition in [4], Theorem 3.1. was formulated mainly for this reason.

Given a compact subset $K$ of $X$ put $Y := X \setminus K$ and define

$$
\mathcal{F}_Y := \{ u \in \mathcal{F} : \tilde{u} = 0 \text{ q.e. on } K \}.
$$

If we restrict $\mathcal{E}$ to the domain $\mathcal{F}_Y$ we obtain again a regular Dirichlet form with associated selfadjoint operator $H_Y \geq 0$. The spectrum of $H_Y$ is discrete by the minmax principle. If $\lambda, \lambda^Y$ are the ground-state energies of $H, H^Y$ one result in [4] may then be written as
Theorem 1. Let \((\mathcal{E}, \mathcal{F})\) be a Dirichlet form satisfying (a)-(d). Then there are constants \(c_1, c_2 \in (0, \infty)\) such that
\[\lambda^Y - \lambda \leq c_1 \text{cap}(K)\]
for all compact subsets \(K\) of \(X\) with capacity smaller than \(c_2\).

Remark 2. Under a more incisive hypothesis it is shown that one can also obtain a lower bound for the ground-state shift
\[\lambda^Y - \lambda \geq c_3 \text{cap}(K)\]
as well as an upper bound for the higher eigenvalues \(\lambda_k, \lambda^Y_k\) of \(H, H^Y\) resp.
\[\lambda^Y_k - \lambda_k \leq c_4 \text{cap}(K).\]
An example in [3] shows that higher eigenvalues need not be shifted, even if the capacity of \(K\) is positive, so a lower bound for the higher eigenvalues cannot be proved without introducing extra conditions.

2. Transference to weighted \(L^2\)-spaces. We consider the unitary transformation
\[U : L^2(X, m) \to L^2(X; \phi^2 m)\]
\[f \mapsto \phi^{-1} f\]
and obtain a new operator
\[H' := UHU^{-1}\]
satisfying \(\sigma(H') = \sigma(H)\). Let \((\mathcal{E}', \mathcal{F}')\) be the closed form associated to \(H'\). It is easy to see that \((\mathcal{E}', \mathcal{F}')\) is again a Dirichlet form, \(\mathcal{F}' = \{\phi^{-1} u : u \in \mathcal{F}\}\) and \(\mathcal{E}'(\phi^{-1} u, \phi^{-1} v) = \mathcal{E}(u, v)\) for \(u, v \in \mathcal{F}\). If we denote by \(\text{cap}'(U)\) the \(\mathcal{E}'\)-capacity of some open set \(U \subset X\), we obtain
\[\text{cap}'(U) = \inf\{\mathcal{E}'(u', u') : u' \in \mathcal{F}', u' \geq 1 \text{ m-a.e. on } U\}\]
\[= \inf\{\mathcal{E}'(\phi^{-1} u, \phi^{-1} u) : u \in \mathcal{F} : u \geq \phi \text{ m-a.e. on } U\}\]
\[= \inf\{\mathcal{E}(u, u) : u \in \mathcal{F} : u \geq \phi \text{ m-a.e. on } U\}\]
\[\leq \text{cap}(U),\]
if \(U\) is such that \(\phi(x) \leq 1 \text{ m-a.e. on } U\). These calculations carry over to arbitrary \(A \subset X\). Therefore we can hope that this new capacity \(\text{cap}'(\cdot)\) yields a
better estimate for the ground-state shift \( \lambda^Y - \lambda \). It would clearly be easiest to prove that the conditions (a)-(d) are fulfilled by \( \mathcal{E}' \) and then use McGillivray’s result. But unfortunately this is not true. Although \( \mathcal{E}' \) is again regular (at least if the ground-state \( \phi \) is continuous, which is not a serious restriction), irreducible and \( H' \) possesses purely discrete spectrum (see Theorem 3 below), the transformed semigroup \( e^{-tH'} \) need not be ultracontractive, so assumption (d) may be violated. The harmonic oscillator provides a counterexample (cf. [1], Section 4.3). There are two possibilities to circumvent this problem. Firstly one could simply require ultracontractivity for \( e^{-tH'} \) and not for \( e^{-tH} \). Since ultracontractivity of these two semigroups are quite independent properties, this would not be a more or less restrictive assumption. We will not discuss this idea further. Another possibility is to take a closer look at the proof of McGillivray’s result. Then it becomes clear that ultracontractivity is only needed to obtain boundedness of \( \phi \) which yields an estimate for \( \int_X \phi d\mu \) for a certain unique measure \( \mu \). We show that this estimate is automatically fulfilled for the transformed form, so we obtain the inequality

\[ \lambda^Y - \lambda \leq c \operatorname{cap}'(K) \]

without ultracontractivity. This is the content of Theorem 6.

**Theorem 3.** Let \((\mathcal{E}, \mathcal{F})\) be a Dirichlet form satisfying conditions (a)-(c) and suppose that the ground-state \( \phi \) is continuous. Then \((\mathcal{E}', \mathcal{F}')\) satisfies (a)-(c) too.

**Proof of Theorem 3.** (b) is trivial because \( U \) is unitary; the proof of (c) is straightforward, so let us concentrate on (a). We have to show the denseness of \( C_0(X) \cap \mathcal{F}' \) in \( \mathcal{F}' \) as well as in \((C_0(X), \| \cdot \|_\infty)\). The first statement is again straightforward, so we will only prove that \( C_0(X) \cap \mathcal{F}' \) is dense in \( C_0(X) \) with respect to uniform norm. Let \( g \in C_0(X) \) arbitrary. Because of the continuity of \( \phi \) we conclude \( h := \phi g \in C_0(X) \), and the regularity of \( \mathcal{E} \) ensures the existence of a sequence \( \varphi_n \in \mathcal{F} \cap C_0(X) \) such that

\[ \varphi_n \to h \text{ in } C_0(X). \]

Urysohn’s theorem (cf. [5], Theorem 2.1.2) implies the existence of \( \eta \in C_0(X) \) with the following property

\[ \eta \equiv 1 \text{ on supp}(h). \]

Now we use again regularity of \( \mathcal{E} \) to find for arbitrary \( \varepsilon > 0 \) a function \( \eta' \in C_0(X) \cap \mathcal{F} \) satisfying

\[ \| \eta - \eta' \|_\infty < \varepsilon. \]
Put

\[ \psi_n := \eta' \phi^{-1} \varphi_n. \]

The proof is completed if we show

(9) \[ \psi_n \in \mathcal{F}' \cap C_0(X) \]

and

(10) \[ \psi_n \to g \text{ in } C_0(X). \]

Proof of (9): Obviously we have \( \psi_n \in C_0(X) \). Because of

\[ \mathcal{F}' = \{ \phi^{-1} u : u \in \mathcal{F} \} \]

it suffices to show that \( \eta' \varphi_n \in \mathcal{F} \). This is a consequence of \( \eta', \varphi_n \in \mathcal{F} \cap L^\infty(X, m) \) and Theorem 1.4.2 of [2]. It remains to prove (10), but this is easy:

\[
\begin{align*}
\| \psi_n - g \|_\infty &= \| \eta' \phi^{-1} \varphi_n - \phi^{-1} h \|_\infty \\
&= \| \eta' \phi^{-1} \varphi_n - \eta \phi^{-1} h \|_\infty \\
&\leq \| \eta' \phi^{-1} \varphi_n - \eta' \phi^{-1} h \|_\infty + \| \eta' \phi^{-1} h - \eta \phi^{-1} h \|_\infty \\
&\leq \| \eta \phi^{-1} \|_\infty \| \varphi_n - h \|_\infty + \| \phi^{-1} h \|_\infty \| \eta' - \eta \|_\infty \\
&\leq 2\varepsilon \| \phi^{-1} h \|_\infty
\end{align*}
\]

if \( n \) is sufficiently large because of (8) and (6), (Note that \( \eta', h \in C_0(X) \) implying \( \| \phi^{-1} h \|_\infty < \infty \).

\[ \square \]

**Definition 4.** A positive measure \( \mu \) is said to be of finite energy integral, if there is a constant \( \gamma \geq 0 \) with

(11) \[
\int_X |\tilde{u}| \, d\mu \leq \gamma \sqrt{\mathcal{E}_1(u, u)}
\]

for all \( u \in \mathcal{F} \) and the collection of such measures is denoted \( S_0 \).

Inequality (11) implies that \( u \mapsto \int_X \tilde{u} d\mu \) defines a continuous linear functional on \( (\mathcal{F}, \mathcal{E}_1(\cdot, \cdot)) \). The Riesz representation theorem entails that there exists a function \( U_1 \mu \in \mathcal{F} \), called the \((1\text{-})potential of } \mu, such that

\[ \mathcal{E}_1(U_1 \mu, u) = \int_X \tilde{u} \, d\mu \]

for all \( u \in \mathcal{F} \).

Before we can prove the main result (Theorem 6), we will need a lemma on the connection between regular Dirichlet forms and Hunt processes. Since the statements of this lemma can be found in [2], we will omit the proof.
LEMMA 5. Given a regular Dirichlet form \((\mathcal{E}, \mathcal{F})\) there is a Hunt process \((\Omega, \mathcal{M}, \mathcal{M}_t, X_t, P_x)\) properly associated with \((\mathcal{E}, \mathcal{F})\) such that the following statements hold

(i) For any compact \(K \subset X\) and any \(f \in \mathcal{F}\) there is a unique measure \(\mu^f_K \in S_0\) supported in \(K\) such that

\[
(U_1 \mu^f_K)(x) = \mathbb{E}_x(e^{-\sigma(K)} \tilde{f}(X_{\sigma(K)})),
\]

where \(\sigma(K) := \inf\{t > 0 : X_t \in K\}\).

(ii) (Dynkin’s formula): For \(f \in \mathcal{F}\) we have

\[
(H + 1)^{-1} f = (H^Y + 1)^{-1} f + U_1 \mu^{(H+1)^{-1} f}_K
\]

and

\[
\mathcal{E}_1((H^Y + 1)^{-1} f, U_1 \mu^{(H+1)^{-1} f}_K) = 0.
\]

(iii) The capacity of a compact set \(K \subset X\) may be computed as

\[
cap(K) = \sup\{\mu(K) : \mu \in S_0, \text{supp}(\mu) \subset K, \mu(X) < \infty, \|U_1 \mu\|_\infty \leq 1\}.
\]

\(\square\)

THEOREM 6. Let \((\mathcal{E}, \mathcal{F})\) be a Dirichlet form satisfying conditions (a)-(c) and let \(0 < \delta < \frac{1}{2}\). Suppose that the ground-state \(\phi\) is continuous on \(X\). For every compact subset \(K\) of \(X\) satisfying \(\cap'(K) < \delta\) we have

\[
\lambda^Y - \lambda \leq c_\delta \cap'(K),
\]

where \(c_\delta := \frac{2(\lambda + 1)}{1-2\delta}\).

PROOF OF THEOREM 6. From Theorem 3 we conclude that \((\mathcal{E}', \mathcal{F}')\) is again a Dirichlet form satisfying (a)-(c). Let \((\Omega', \mathcal{M}', \mathcal{M}_t', X_t', P'_x)\) be the process associated with \((\mathcal{E}', \mathcal{F}')\) and define \(S'_0, U'_1 \mu\) in the obvious way. Further let \(\nu := \mu^{(H'+1)^{-1}}_K \in S'_0\) be the unique measure corresponding to \(K\) and \((H' + 1)^{-1} 1(\in \mathcal{F}')\) in the sense of Lemma 5 (i). Because of

\[
\|U_1 \nu\|_\infty = \text{ess sup} \mathbb{E}_x(e^{-\sigma'(K)} \frac{1}{1+\lambda}) \leq 1,
\]

\[
\nu(X) = \int_X 1 d\nu \leq \gamma \mathcal{E}'_1(1, 1) = (\lambda + 1) \gamma < \infty
\]
and Lemma 5 (iii) we see that

\[ \nu(K) \leq \text{cap}'(K). \]

Therefore we obtain

\[
1 + \lambda Y = \inf_{g \in \mathcal{F}_Y} \frac{\mathcal{E}_1'(g, g)}{\|g\|_{L^2(X, \phi^2 \mu)}^2} \\
\leq \frac{\mathcal{E}_1'((H^Y + 1)^{-1}, 1, (H^Y + 1)^{-1}, 1)}{\|((H^Y + 1)^{-1})_{L^2(X, \phi^2 \mu)} - 2((H^Y + 1)^{-1}, U_1' \nu)_{L^2(X, \phi^2 \mu)} \| (\lambda + 1)^{-1}} \\
= \frac{(\lambda + 1)^{-2} - 2(\lambda + 1)^{-1} \mathcal{E}_1'(1, U_1' \nu)}{(\lambda + 1)^{-2} - 2\lambda + 1 \text{cap}'(K)} \\
\leq \frac{\lambda + 1}{1 - 2\text{cap}'(K)} \\
= \frac{(\lambda + 1)(1 - 2\text{cap}'(K)) + 2(\lambda + 1)\text{cap}'(K)}{1 - 2\text{cap}'(K)} \\
= 1 + \lambda + \frac{2(\lambda + 1)\text{cap}'(K)}{1 - 2\text{cap}'(K)} \\
\leq 1 + \lambda + c_5 \text{cap}'(K),
\]

finishing the proof of the theorem.

\[ \square \]

References


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