ON CRITICAL AND SUBCRITICAL SEMIGROUPS

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Abstract. The theory of transient and recurrent submarkovian semigroups and its spectral-theoretic counterpart is extended to the case of subcritical and critical semigroups under certain conditions.

0. Introduction. Roughly speaking, a non-negative definite Schrödinger operator is said to be critical if the bottom of the spectrum jumps to the left whenever we subtract a non-negative non-trivial potential, and is said to be subcritical otherwise. This phenomenon is mirrored in the behaviour of the associated Green operator, see [5], [6], [7], [8], [11], [12], amongst others. Transient resp. recurrent submarkovian semigroups are special cases of subcritical resp. critical semigroups. The former have been studied in [3] in terms of the behaviour of the Green function, in the context of Dirichlet forms. A spectral-theoretic interpretation was given in [4] in an abstract setting. The aim of the present article is to extend the results of [3] and [4] to the case of critical and subcritical semigroups in the context of Dirichlet forms, which we manage under certain conditions. The various results and their proofs contained in the next Section up to Theorem 2.12 therefore follow closely [3], while the spectral-theoretic part, Theorems 2.12 and 2.13 is based on [4].

1. Critical and subcritical semigroups. Let $X$ be a locally compact separable metric space and $m$ an everywhere positive Radon measure on the Borel $\sigma$-algebra $\mathcal{B}$ on $X$. A strongly continuous symmetric markovian semigroup $(T_t)_{t>0}$ on $L^2(X,m)$ is given. Denote by $-H$ its generator. We occasionaly refer to the resolvent $G_\alpha := (H + \alpha)^{-1}$. We assume

(A.1) $(T_t)_{t>0}$ is ultracontractive, i.e. $\|T_t\|_1,\infty < \infty \forall t > 0$. 
Denote by \((F, \mathcal{E})\) the Dirichlet form associated to \((T_t)_{t>0}\). We suppose that this is regular. Let \(V = V_+ - V_-\) be a measurable function such that \(V_+ \in L^1_{loc}(X, m)\) and \(V_-\) lies in the Kato class \(V_- \in S_K\) (cf. [10]). The form

\[
F^V := \{u \in F : V_+ u \in L^2(X, m)\},
\]

\[
\mathcal{E}^V(u, v) := \mathcal{E}(u, v) + \int_X uv \, V_+ \cdot dm - \int_X uv \, V_- \cdot dm; \ u, v \in F^V
\]

is a closed form. The corresponding strongly continuous \(L^2\)-semigroup is denoted by \((T^V_t)_{t>0}\) and has generator \(-H^V\). The semigroup \((T^V_t)_{t>0}\) is again ultracontractive ([10], theorem 5.1), so by the Dunford-Pettis theorem (cf. [9], corollary A.1.2, for example) for each \(t > 0\) there exists a measurable function \(p^V_t : X \times X \to [0, \infty)\) such that

\[
T^V_t f(x) = \int_X p^V_t(x, y) \, f(y) \, m(dy) \, m\text{-a.e. } \forall f \in L^1(X, m).
\]

Since \((T^V_t)_{t>0}\) acts as a \(C_0\)-semigroup in \(L^1(X, m)\) ([10], corollary 4.2) it makes sense to define

\[
S^V_t f(x) := \int_0^t T^V_s f(x) \, ds, \ f \in L^1(X, m)
\]

and

\[
G^V f(x) := \lim_{t \to \infty} S^V_t f(x), \ f \in L^1_+(X, m).
\]

Define the class of \((p^V_t)_{t>0}\)-supermedian functions by

\[
S := \{\phi : X \to \mathbb{R} \mid \phi \text{ is measurable, } 0 < \phi < \infty \text{ m-a.e.,} \}
\]

\[
p^V_t \phi \leq \phi \text{ m-a.e., } \forall t > 0
\]

We assume

\[(A.2) \ S \neq \emptyset \text{ and } H^V \geq 0.\]

We shall show later that \((A.2)\) holds for Schrödinger operators on \(\mathbb{R}^d\), using the Allegretto-Piepenbrink theorem.

The next Lemma is a slight generalisation of Hopf's maximal ergodic inequality (cf. [3]).
LEMMA 2.1. Let $Y$ be a measurable $(T_t^V)_{t>0}$-invariant subset of $X$ and $\phi \in S$. Given $f \in L^1(X,m) \cap L^1(X,\phi \cdot m)$ and $h > 0$ let $E_h := \{x \in Y : \sup_n S^V_{nh} f(x) > 0\}$. Then
\[
\infty > \int_{E_h} S^V_h f \phi \cdot dm \geq 0.
\]

PROOF. Let
\[
E^n_h := \{x \in Y : \max_{1 \leq k \leq n} S^V_{k h} f(x) > 0\} = \{x \in Y : \max_{1 \leq k \leq n} (S^V_{k h} f)^+(x) > 0\}.
\]

Then for $m$-a.e. $x \in E^n_h$,
\[
S^V_h f(x) + \max_{1 \leq k \leq n} (S^V_{(k+1) h} f - S^V_h f)^+(x) \geq \max_{1 \leq k \leq n} (S^V_{k h} f)^+(x).
\]

Since $(T_t^V)_{t>0}$ is positivity-preserving,
\[
\max_{1 \leq k \leq n} (S^V_{(k+1) h} f - S^V_h f)^+(x) \leq T^V_h (\max_{1 \leq k \leq n} (S^V_{k h} f)^+)(x).
\]

Thus,
\[
\int_{E^n_h} S^V_h f \phi \cdot dm
\]
\[
\geq \int_{E^n_h} \{ \max_{1 \leq k \leq n} (S^V_{k h} f)^- - p_t^Y (\max_{1 \leq k \leq n} (S^V_{k h} f)^+) \} \phi \cdot dm
\]
\[
\geq \int_{E^n_h} \max_{1 \leq k \leq n} (S^V_{k h} f)^+ \phi \cdot dm - \int_X \max_{1 \leq k \leq n} (S^V_{k h} f)^+ p_t^Y (1_{E^n_h} \phi) dm
\]

where the last line follows by Fubini's theorem, and the symmetry of $T_t^V$,
\[
\geq \int_{E^n_h} \max_{1 \leq k \leq n} (S^V_{k h} f)^+ \phi \cdot dm - \int_Y \max_{1 \leq k \leq n} (S^V_{k h} f)^+ \phi \cdot dm = 0
\]

where in the last line we have used the invariance of the set $Y$, i.e.
\[
p_t^V (1_{E^n_h} \phi) = 1_Y p_t^V (1_{E^n_h} \phi) \leq 1_Y p_t^V \phi \leq 1_Y \phi,
\]

$m$-a.e. The result now follows on letting $n \to \infty$. 
LEMMA 2.2. Let $\phi \in S$. Let $g \in L^1(X, m) \cap L^1(X, \phi \cdot m)$ be strictly positive m.a.e. and $Y$ a $(T_t^V)_{t > 0}$-invariant subset of $X$ on which $G^V g < \infty$ m.a.e. Then $G^V f < \infty$ m.a.e. on $Y$ for all $f \in L^1(X, m) \cap L^1(X, \phi \cdot m)$, $f \geq 0$ m.a.e.

PROOF. Given $f \in L^1(X, m) \cap L^1(X, \phi \cdot m)$, $f \geq 0$ m.a.e. and $a, h > 0$ put

$$A := \{ x \in Y : \sup_n S^V_{nh}(f - ag)(x) > 0 \}.$$

By Lemma 2.1,

$$\infty > \int_A S^V_h (f - ag) \phi \cdot dm \geq 0.$$

Note that the set $B := \{ x \in Y : G^V f(x) = \infty \}$ is contained in the set $A$ up to an $m$-negligible set. So we can write

$$h \int_X f \phi \cdot dm \geq \int_X S^V_h f \phi \cdot dm \geq \int_A S^V_h f \phi \cdot dm \geq a \int_A S^V_h \phi \cdot dm,$$

and consequently: $\frac{1}{a} \int_X f \phi \cdot dm \geq \frac{1}{h} \int_B S^V_h g (\phi \wedge N) \cdot dm$ for any $N \in \mathbb{N}$. The strong continuity of $(T_t^V)_{t > 0}$ on $L^1(X, m)$ implies the $L^1$-convergence of $\frac{1}{h} S^V_h g$ to $g$ in the limit $h \downarrow 0$. This leads to the inequality $\frac{1}{a} \int_X f \phi \cdot dm \geq \int_B g (\phi \wedge N) \cdot dm$. On letting $N \to \infty$ and then $a \to \infty$ we see that $\int_B g \phi \cdot dm = 0$, from which follows $m(B) = 0$.

LEMMA 2.3. Let $\phi$ and $g$ be as in the last Lemma. Then the sets $\{ x \in X : G^V g(x) = \infty \}$ and $\{ x \in X : G^V g(x) = 0 \}$ are $(T_t^V)_{t > 0}$-invariant. Moreover, these sets are m.a.e. independent of the choice of the strictly positive function $g \in L^1(X, m) \cap \bigcup_{\phi \in S} L^1(X, \phi \cdot m)$.

PROOF. For $g$ as in the Lemma, put $B_n := \{ x \in X : G^V g(x) \leq n \}$ and $B := \{ x \in X : G^V g(x) < \infty \}$. Then for any $f \in L^1(X, m)$,

$$(T_t^V(1_{B_n} f), G^V g) = (1_{B_n} f, T_t^V G^V g) \leq n(f, 1).$$

This means that $T_t^V(1_{B_n} f) = 0$ on $X \setminus B$. Letting $n \to \infty$ we get that $T_t^V(1_B f) = 0$ m.a.e. on $X \setminus B$. This shows that $B$ is $(T_t^V)$-invariant. The invariance of the set $\{ x \in X : G^V g(x) = 0 \}$ follows by a similar argument.

Suppose now that $\phi_i \in S$ and $g_i \in L^1(X, m) \cap L^1(X, \phi_i \cdot m)$, $i = 1, 2$ are strictly positive. Set $B_i := \{ x \in X : G^V g_i(x) < \infty \}$. Then $B_1 \cup B_2$ is $(T_t^V)$-invariant, $G^V (g_1 \wedge g_2) < \infty$ m.a.e. on $B_1 \cup B_2$ and $g_1 \wedge g_2 \in L^1(X, m) \cap L^1(X, \phi_1 \cdot m) \cap L^1(X, \phi_2 \cdot m)$. By Lemma 2.2, we conclude that $G^V f < \infty$ m.a.e. on $B_1 \cup B_2$ for all $f \in L^1(X, m) \cap \bigcup L^1(X, \phi_i \cdot m)$. In particular,
\[ G^V g_1 < \infty \text{ m.-a.e. on } B_1 \cup B_2 \text{ and the same holds for } g_2. \text{ Thus } B_1 = B_2 \text{ up to an m-negligible set.} \]

**Definition 2.4.** We say that the semigroup \((T_t^V)_{t>0}\) is subcritical if \(G^V f < \infty\) m.-a.e. for all \(f \in L^1(X, \mu) \cap \bigcup_{\phi \in S} L^1(X, \phi \cdot m), f \geq 0\) m.-a.e..

**Lemma 2.5.** The semigroup \((T_t^V)_{t>0}\) is subcritical if and only if for some \(\phi \in S\) and strictly positive \(g \in L^1(X, \mu) \cap L^1(X, \phi \cdot m), G^V g < \infty\) m.-a.e..

**Proof.** The necessity is clear. Let \(\phi' \in S\) and \(f \in L^1(X, \mu) \cap L^1(X, \phi' \cdot m), f \geq 0\) m.-a.e.. Choose a strictly positive function \(g' \in L^1(X, \mu) \cap L^1(X, \phi' \cdot m).\)

By Lemma 2.3, \(\{x \in X : G^V g'(x) < \infty\} = \{x \in X : G^V g(x) < \infty\} = X\) up to an m-negligible set. It follows now from Lemma 2.2 that \(G^V f < \infty\) m.-a.e. on \(X\).

**Definition 2.6.** For a fixed strictly positive \(g \in L^1(X, \mu) \cap L^1(X, \phi \cdot m)\) for some \(\phi \in S\), we call the invariant sets \(X_d := \{x \in X : G^V g(x) < \infty\}\) resp. \(X_c := \{x \in X : G^V g(x) = \infty\}\) the dissipative resp. conservative parts of \(X\) relative to \((T_t^V)\).

**Lemma 2.7.** Suppose that \(f \in L^1(X, \mu) \cap L^1(X, \phi \cdot m)\) for some \(\phi \in S\). Set \(B := \{x \in X_c : G^V f(x) < \infty\}\). If \(m(B) > 0\) then \(f = 0\) and \(G^V f = 0\) m.-a.e. on \(B\).

**Proof.** In view of Lemma 2.3 we can suppose that \(X_c = \{x \in X : G^V g(x) = \infty\}\) where \(g\) is a strictly positive function belonging to \(L^1(X, \mu) \cap L^1(X, \phi \cdot m)\) with \(\phi\) as in the Lemma. Since \(G^V g(x) = \infty\) m.-a.e. on \(X_c\) the set \(B\) is contained in the set \(A := \{x \in X : \sup_n S_{nh}^V (g - af)(x) > 0\}\) up to an m-negligible set for any \(a, h > 0\). Exactly as in the proof of Lemma 2.2, we obtain the inequality \(\frac{1}{a} \int_x g \phi \cdot dm \geq \frac{1}{h} \int_B S_h^V f \phi \cdot dm\). On letting \(a \to \infty\) we get that \(\int_B S_h^V f \phi \cdot dm = 0\) for any \(h > 0\), and hence \(\int_B G^V f \phi \cdot dm = 0\) by Fatou's lemma. On the other hand, since \(\frac{1}{h} S_h^V f\) converges to \(f\) in \(L^1(X, \mu)\) as \(h \to 0\), we have \(\frac{1}{a} \int_x g \phi \cdot dm \geq \int_B f \phi \cdot dm\), from which follows \(\int_B f \phi \cdot dm = 0\). Thus both \(f\) and \(G^V f\) are m.-a.e. equal to 0 on \(B\).

**Definition 2.8.** We say that the semigroup \((T_t^V)_{t>0}\) is critical if \(G^V f(x) = 0\) or \(\infty\) m.-a.e. for all \(f \in L^1(X, \mu) \cap \bigcup_{\phi \in S} L^1(X, \phi \cdot m), f \geq 0\) m.-a.e..

**Lemma 2.9.**

(i.) \((T_t^V)_{t>0}\) is subcritical if and only if \(X_d = X\) m.-a.e..

(ii.) \((T_t^V)_{t>0}\) is critical if and only if \(X_c = X\) m.-a.e..

(iii.) Suppose that \((T_t^V)_{t>0}\) is irreducible. Then it is either subcritical or critical. In the critical case, \(G^V f = \infty\) m.-a.e. on \(X\) for any non-negative measurable function \(f\) such that \(m(\{x \in X : f(x) > 0\}) > 0\).
PROOF. (i.) is clear from Lemma 2.5. For (ii.), first suppose that \((T_t^V)_{t>0}\) is critical. Let \(g\) be a strictly positive function such that \(g \in L^1(X, m) \cap L^1(X, \phi \cdot m)\) for some \(\phi \in \mathcal{S}\). Then \(G^V g\) takes only the values 0 and \(+\infty\) by definition. Let \(A\) be the invariant set given by \(A := \{x \in X : G^V g(x) = 0\}\) (cf. Lemma 2.3) and suppose that \(A\) has positive measure. Now, for any \(\alpha > 0\),

\[
0 = 1_A G^V g = G^V 1_A g \geq G^V_\alpha 1_A g.
\]

So \(A\) must have measure zero; hence \(X_c = X\). Conversely, suppose that \(X_c = X\) m-a.e.. Then by Lemma 2.7, \(G^V f\) may only take the values 0 or \(\infty\) for all \(f \in L^1(X, m) \cap \bigcup_{\phi \in \mathcal{S}} L^1(X, \phi \cdot m)\). In other words, \((T_t^V)_{t>0}\) is critical. We now turn to the proof of (iii.). Since \(X_c, X_d\) are \((T_t^V)\)-invariant, disjoint and cover \(X\), \((T_t^V)\) is either critical or subcritical. To see the second statement of (iii.), if \(A := \{x \in X : G^V f(x) < \infty\}\) has positive measure, then \(A = X\) up to an \(m\)-negligible set by irreducibility. But then Lemma 2.7 entails that \(f = 0\) m-a.e. on \(X\).

**Lemma 2.10.** For any non-negative function \(g \in L^1(X, m) \cap L^2(X, m)\),

\[
\sup_{u \in \mathcal{F}^V} \frac{(|u|, g)}{\mathcal{E}^V(u, u)} = \sqrt{\int_X g \, G^V g \, dm} \leq +\infty.
\]

**Proof.** The proof is the same as in [3], lemma 1.5.3.

**Definition 2.11.** We say that a non-negative definite quadratic form \((\mathcal{E}^V, \mathcal{F}^V)\) is subcritical if there exists a bounded \(m\)-integrable function \(g\) strictly positive \(m\)-a.e. on \(X\) such that

\[
\int_X |u| g \, dm \leq \sqrt{\mathcal{E}^V(u, u)} \quad \forall u \in \mathcal{F}^V.
\]

**Theorem 2.12.** \((T_t^V)_{t>0}\) is subcritical if and only if \((\mathcal{E}^V, \mathcal{F}^V)\) is.

**Proof.** Suppose that \((T_t^V)_{t>0}\) is subcritical. Choose a strictly positive bounded measurable function \(f \in L^1(X, m) \cap L^1(X, \phi \cdot m)\) for some \(\phi \in \mathcal{S}\), satisfying \(\int_X f \, dm = 1\). Since \(G^V f(x) < \infty\) m-a.e., we can define \(g := f/(G^V f \vee 1)\). Then \(g \leq f\) m-a.e., \(g\) is m-a.e. strictly positive, and \(G^V g \leq 1\) m-a.e.. In particular, \(\int_X g \, G^V g \, dm \leq 1\). Lemma 2.10 now implies the subcriticality of the form \((\mathcal{E}^V, \mathcal{F}^V)\). The converse assertion follows immediately from Lemma 2.5.
We now describe a spectral-theoretic interpretation of the notions of criticality and subcriticality. We first of all introduce some notation. Let $H$ be a non-negative definite self-adjoint operator on $L^2(X, m)$. A non-negative measurable function $W$ is said to be $H$-admissible if $H - W$ is closable and the closure, again denoted by $H - W$ is such that $-H + W$ generates a strongly continuous $L^2$-semigroup. The notation $s(H)$ stands for the spectral bound of $H$, $s(H) \equiv \inf \{\lambda \in \mathbb{R} : \lambda \in \sigma(H)\}$.

**Theorem 2.13.** Assume that $(T_t^V)_{t>0}$ is irreducible. If $(T_t^V)_{t>0}$ is critical, then $s(H^V - W) < 0$ for all $H^V$-admissible non-trivial $W \geq 0$.

**Proof.** Let $W$ be a non-negative non-trivial measurable function which is $H^V$-admissible. Replacing $W$ by $(W \wedge c)1_K$ for an appropriate constant $0 < c < \infty$ and $K \in \mathcal{B}$ with $m(K) < \infty$, we may suppose that $W \in L^2(X, m) \cap L^\infty(X, m)$. Fix $u_0 \in \mathcal{F}^V$ such that $u_0 > 0$ m-a.e.. Then for all $u \in \mathcal{F}^V$, $(Wu_0, u)^2 \leq c^{1/2}(Wu_0, u)^{1/2}$ where $c := (Wu_0, u_0)$. Suppose for a contradiction that $s(H^V - W) \geq 0$. This implies that $(Wu_0, u)^2 \leq c^{1/2}E^V(u, u)^{1/2}$ for all $u \in \mathcal{F}^V$. Moreover,

$$(Wu_0, S_t^VWu_0)^2 \leq cE^V(S_t^VWu_0, S_t^VWu_0)$$

$$= c(Wu_0 - T_t^VWu_0, S_t^VWu_0)$$

$$\leq c(Wu_0, S_t^VWu_0).$$

In other words, $(Wu_0, S_t^VWu_0) \leq c$ for all $t > 0$. Let $\phi \in \mathcal{S}$. Multiplying $W$ by $1_{\phi \leq n}$ for appropriate $n$ if necessary we can suppose that $Wu_0$ belongs to $L^1(X, m) \cap L^1(X, \phi \cdot m)$. By Fatou's lemma, the above inequality then entails that $G^VWu_0 < \infty$ m-a.e.. Subcriticality of $(T_t^V)_{t>0}$ now follows from Lemma 2.9, providing the desired contradiction.

**Theorem 2.14.** Let $\{K_n\}$ be an exhaustion of $X$ by sets of finite measure. Assume that $1_{K_n}G_1^V$ is compact for each $n \in \mathbb{N}$. If $s(H^V - W) < 0$ for each non-trivial non-negative $H^V$-admissible $W$, then $(T_t^V)_{t>0}$ is critical.

**Proof.** For $W \in L^\infty(X, m)$ supported in some $K_n$ the resolvent difference $G_1^V - G_1^{V-W}$ is compact; hence by Weyl's theorem, the operators $H^V, H^V - W$ share the same essential spectrum. Since $s(H^V - W) < 0$, while $s(H^V) \geq 0$, there must exist $\phi_W \in L^2(X, m)$, $\phi_W > 0$ satisfying $(H^V - W)\phi_W = s(H^V - W)\phi_W$. The same argument for $\epsilon W, \epsilon > 0$ yields the existence of a strictly positive $L^2$-function $\phi_{\epsilon W}$ such that

$$(H^V - \epsilon W)\phi_{\epsilon W} = s(H^V - \epsilon W)\phi_{\epsilon W}.$$
This can be rearranged to get \( \phi_{\epsilon W} = (H^V - s(H^V - \epsilon W))^{-1}\epsilon W\phi_{\epsilon W} \), and thus
\[
\epsilon W^{1/2}(H^V - s(H^V - \epsilon W))^{-1}W^{1/2}(W^{1/2}\phi_{\epsilon W}) = W^{1/2}\phi_{\epsilon W}.
\]
Taking the \( L^2 \)-operator norm we get
\[
1 \leq \epsilon\|W^{1/2}(H^V - s(H^V - \epsilon W))^{-1}W^{1/2}\|_{L^2}
\]
Suppose for a contradiction that \( (T^V_t)_{t>0} \) is subcritical. Then there exists a strictly positive bounded measurable function \( g \) in \( L^1(X, m) \cap L^1(X, \phi \cdot m) \) for \( \phi \in S \) such that \( gG^V g \in L^1(X, m) \cap L^\infty(X, m) \) (cf. the proof of Theorem 2.12). Using the above estimate for the potential \( W = 1_{K_n}g^2 \) where \( n \) is large enough so that \( W \) is non-trivial yields
\[
1 \leq \epsilon\|1_{K_n}g(H^V - s(H^V - \epsilon 1_{K_n}g^2))^{-1}1_{K_n}g\|_{L^2} \leq \epsilon\|1_{K_n}g G^V 1_{K_n}g\|_{L^\infty}
\]
for all \( \epsilon > 0 \) by Riesz-Thorin interpolation ([2], 1.1.5). Letting \( \epsilon \to 0 \) gives the desired contradiction.

**Example 2.15.** We now describe an example where the above theory applies. Let \( H := -\frac{1}{2}\Delta \) acting in \( L^2(\mathbb{R}^d, dx) \). Let \( V = V_+ - V_- \) with \( V_+ \in K_d \) and \( V_- \in K_{d,loc} \), the Kato and local Kato classes in \( \mathbb{R}^d \). Assume that \( H^V = H + V \) is non-negative definite. Condition (A.1) holds due to [10]. We now verify the remainder of (A.2). The following is a variation of the construction in [1], Theorem 2.12. Let \( \{f_n\} \) be a sequence of \( C^\infty_0 \) functions which are non-negative and supported in \( \{x \in \mathbb{R}^d : n \leq |x| \leq 2n\} \). Let \( \phi_n := c_n G^V_{n-1}f_n \) where \( c_n \) is chosen such that \( \phi_n(0) = 1 \). By Harnack’s inequality we can find for each \( R > 0 \) a constant \( C_R > 0 \) such that
\[
C_R^{-1} \leq \phi_n(x) \leq C_R \text{ if } |x| < R
\]
for all large \( n \). Define \( \phi := \liminf_{n \to \infty} \phi_n \). Then clearly \( 0 < \phi < \infty \) \( dx \)-a.e.. Note that \( p^V_t \phi_n \leq e^{n^{-1}t} \phi_n \). By Fatou’s lemma,
\[
p^V_t \phi(x) = \int_{\mathbb{R}^d} p^V_t(x, y) \liminf_{n \to \infty} \phi_n(y) dy \leq \liminf_{n \to \infty} p^V_t \phi_n(x) \leq \liminf_{n \to \infty} e^{n^{-1}t} \phi_n(x) = \phi(x).
\]
Thus \( S \neq \emptyset \).
References


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