REMARKS ON REGULAR DEPENDENCE
OF LIMIT SETS AND PROLONGATIONS ON POINTS
IN DYNAMICAL SYSTEMS

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0. Introduction. The purpose of the present paper is to discuss upper semi-continuity of limit sets, prolongational limit sets and prolongations in dynamical systems on metric spaces. We shall consider two kinds of upper semi-continuity conditions (Heine and Cauchy types). There are some essential differences between conditions sufficient for these two upper semi-continuities discussed with respect to mappings associating with a given point \( x \) the sets \( \Lambda^+(x) \), \( J^+(x) \), \( D^+(x) \) etc. (definitions are recalled below), relatively to the properties of the space in which these sets are considered and to the properties of that sets themselves, as well as on the stability type properties of motions being under investigations.

The paper extends certain results of [6] and gives some remarks complementary to the results of [7] where regularity (Cauchy upper and lower semi-continuity) of mapping associating with a given point \( x \) the sets \( \pi_+(x) \), \( \Lambda^+(x) \), \( J^+(x) \) and \( D^+(x) \) is discussed (mainly under the assumption that \( \pi_+(x) \) is compact).

1. Preliminaries. We shall discuss dynamical systems on metric spaces, in particular on finite dimensional Euclidean spaces. In order to exclude any misfit, we recall fundamental definitions and the usual terminology (compare for instance [1], [2]).

Let \( (X, \rho) \) be a metric space. We say that the triplet \( (X, \mathbb{R}; \pi) \) where \( \pi \) is a mapping from \( \mathbb{R} \times X \) into \( X \) is a dynamical system if:

\[ \pi \text{ is continuous and} \]

\[ (1.1) \quad \pi(0, x) = x, \quad \pi(t, \pi(s, x)) = \pi(t + s, x) \quad \text{for} \quad x \in X, \ s, t \in \mathbb{R}. \]
Let \( x \in X \) be given. The mapping

\[
\pi^x : \mathbb{R} \ni t \rightarrow \pi^x(t) := \pi(t, x) \in X
\]

is called the motion of \( x \). The motion \( \pi^x \) is said to be positively (negatively) stable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\rho(x, y) < \delta \implies \rho(\pi(t, x), \pi(t, y)) < \varepsilon \quad \text{for} \quad t \geq 0 \ (t \leq 0).
\]

**Remark 1.1.** Stability of motions can be generalized by introducing the so-called semi-stability. The motion \( \pi^x \) is said to be positively (negatively) semi-stable if for every \( \varepsilon > 0 \) there exist \( \delta > 0 \) and \( t^0 \geq 0 \ (t^0 \leq 0) \) such that

\[
\rho(x, y) < \delta \implies \rho(\pi(t, x), \pi(t, y)) < \varepsilon \quad \text{for} \quad t \geq t^0 \ (t \leq t^0).
\]

(compare [5]).

**Remark 1.2.** It is easy to observe that in the case of systems considered in the present paper, in which \( \pi \) is continuous, the conditions of stability and semi-stability are equivalent. However, if we admit systems with \( \pi \) not necessary continuous (called by the author in other papers pseudo-dynamical systems, cf. for instance [6]), then in general, the stability conditions are essentially stronger than the corresponding semistability ones.

Now let us recall further definitions which we will need in the sequel.

For \( x \in X \) we denote by \( \pi(x), \pi_+(x) \) and \( \pi_-(x) \) the sets

\[
\pi^x(\mathbb{R}) = \{\pi(t, x) : t \in \mathbb{R}\}, \quad \pi^x([0, \infty)) \quad \text{and} \quad \pi^x((-\infty, 0])
\]

respectively and we call them: the trajectory of \( x \), the positive semi-trajectory of \( x \) and the negative semi-trajectory of \( x \). A point \( x \) is said to be a stationary point if \( \pi(x) = \{x\} \) (this is equivalent to \( \pi_+(x) = \{x\} \) as well as to \( \pi_-(x) = \{x\} \)).

If \( A \subset X \) then

\[
\pi(A) = \bigcup \{\pi(x) : x \in A\}, \\
\pi_+(A) = \bigcup \{\pi_+(x) : x \in A\}, \\
\pi_-(A) = \bigcup \{\pi_-(x) : x \in A\}.
\]

For \( x \in X \) we define the positive limit set for \( x \), \( \Lambda^+(x) \) by the formula:

\[
\Lambda^+(x) := \{y : \text{there is a sequence} \{t_n\} \text{ of real numbers} \\
\text{such that} \ t_n \rightarrow \infty \text{ and} \ \pi(t_n, x) \rightarrow y \text{ as} \ n \rightarrow \infty\}.
\]
The negative limit set $\Lambda^-(x)$ for $x$ is defined by substituting: \( t_n \to -\infty \)

in the place of \( t_n \to \infty \) in the definition (1.4).

Furthermore we put

\[
J^+(x) := \{ y : \text{there is a sequence } \{t_n\} \text{ of real numbers and } \{x_n\} \text{ of elements of } X \text{ such that } t_n \to \infty, x_n \to x \text{ and } \pi(t_n, x_n) \to y \text{ as } n \to \infty \},
\]

and

\[
D^+(x) := \left\{ y \in X : \text{there are sequences } \{t_m\} \text{ of real numbers and } \{y_m\} \text{ of elements of } X \text{ such that } t_m \geq 0, y_m \to x \text{ and } \pi(t_m, y_m) \to y \text{ as } m \to \infty \right\}.
\]

and call these sets: the positive prolongational limit set for $x$ and the (first) positive prolongation of $x$ (see for instance [1], [2]). The sets $J^-(x)$ and $D^-(x)$ are defined in such a way that the condition: \( t_n \to \infty \) is replaced by \( t_n \to -\infty \) in (1.5) and – respectively – the condition \( t_n \geq 0 \) is replaced by \( t_n \leq 0 \) in (1.6).

We shall use in the sequel the classical notation: for $x \in X$ and $A \subset X$ we put

\[
\rho(x, A) := \inf \{ \rho(x, y) : y \in A \}
\]

and for $r > 0$, $A \subset X$

\[
B(A, r) := \{ y \in X : \rho(y, A) < r \}.
\]

If $A = \{ x \}$ then

\[
B(A, r) = B(\{ x \}, r) = \{ y : \rho(x, y) < r \}
\]

is the usual open ball centered at $x$ with the radius $r$ and is of course denoted by $B(x, r)$.

**Remark 1.3.** The definitions of $\Lambda^+(x)$, $J^+(x)$, $D^+(x)$ (and the "symmetric" negative sets) recalled above can be extended in order to obtain corresponding notions in more general systems with the metric space $X$ replaced by general topological spaces. In order to do that one can replace the "sequential" definitions by suitable definitions of the "neighbourhood type" (see for instance [1], [2]); for metric spaces both types of definitions give the same notions and so we have introduced them using (1.4) – (1.6).
Remark 1.4. The definition (1.4) - (1.6) are correctly stated also with respect to systems in which \( \pi \) does not need to be continuous. Moreover, some theorems on regular dependence of sets \( \Lambda^+(x), J^+(x), D^+(x) \) on \( x \) can be proved also for such more general cases (compare Remark 2.2 below).

We shall need some well known properties of the sets (1.4) - (1.6); we are collecting them by formulating the following (see [1], [2] for example)

**Proposition 1.1.** Let \( x \in X \) be given. Then

(i) \( \Lambda^+(x), J^+(x), D^+(x) \) are closed

(ii) \( \pi_+(x) = \pi_+(x) \cup \Lambda^+(x) \)

(iii) \( D^+(x) = J^+(x) \cup \pi_+(x) \).

Similar ("symmetric") conditions are satisfied for \( \Lambda^-(x), J^-(x) \) and \( D^-(x) \).

2. Upper semi-continuity of set-valued mappings. Let \( F \) be a mapping defined in the metric space \( X \), ranged in the family of all subsets of \( X \). There are considered two conditions of the upper semi-continuity of \( F \) at a given point \( x \in X \). Under suitable additional assumptions they are equivalent, but we shall consider them with respect to such mappings for which that equivalence will not be assured in general; only implication in one direction will be observed. The conditions introduced below are well known (see [3], [4], [7] for example) and so we do not discuss all details limiting ourselves to semi-continuity of mappings associating with a given point \( x \) the sets \( \Lambda^+(x), J^+(x), D^+(x), \Lambda^-(x) \) etc. in two versions, calling them in this paper semi-continuity in the sense of Heine and Cauchy.

Let \( x \in X \) be given. We say that the mapping

\[
F: X \ni x \longrightarrow F(x) \in 2^X
\]

is (H)-usc (upper semi-continuous in the Heine sense) at the point \( x \) if and only if the following implication holds:

if \( \{x_n\} \) and \( \{y\} \) are sequences of elements of \( X \) such that

\[
x_n \to x, \quad y_n \to y \quad \text{as} \quad n \to \infty
\]

and

\[
y_n \in F(x_n) \quad \text{for every} \quad n,
\]

then

\[
y \in F(x).
\]
We say that $F$ is (C)-usc (upper semi-continuous in the Cauchy sense) at $x$ if and only if the for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(y \in X, \rho(x, y) < \delta) \implies F(y) \subset B(F(x), \varepsilon).$$

**Remark 2.1.** It is clear that if $F(x)$ is closed, then (C)-usc $\iff$ (H)-usc. However, the inverse implication is not true in general without additional assumptions on the space $X$ and the mapping $F$. It is known that if $X$ is locally compact and $F(x)$ is compact then the conditions (H)-usc and (C)-usc are equivalent.

Let us consider now a dynamical system $(X, R, \pi)$ (with $(X, \rho)$ being a metric space) and the mapping

$$(2.1) \quad y \mapsto \Lambda^+(y).$$

It is possible to prove the following

**Theorem 2.1 (Th. 2 in [5]).** If $x \in X$ is such that the motion $\pi^X$ is positively stable then the mapping (2.1) in (H)-usc at the point $x$.

**Remark 2.2.** It is possible to generalize the above theorem as follows: we may drop the continuity of $\pi$ and replace the positive stability of $\pi^x$ by the positive semi-stability of $\pi^x$ (compare Remark 1.2).

**Remark 2.3.** It is clear that replacing positive stability of $\pi^x$ (or – in the more general case – the positive semi-stability of $\pi^x$) by the negative stability (by the negative semi-stability, respectively) we are able to show that the mapping

$$(2.2) \quad y \mapsto \Lambda^-(y)$$

is (H)-usc at the point $x$.

In order to make our paper relatively self-contained and also in order to compare methods of proofs, we will recall here the main idea of the proofs of Theorem 2.1.

Assume that $\pi^x$ is positively stable. Suppose that $\{x_n\}$ and $\{y_n\}$ are such that $x_n \to x$, $y_n \to y$, $y_n \in \Lambda^+(x_n)$. For every $n$ there exists a sequence of real numbers $\{t^m_n\}_{m=1}^\infty$ such that

$$(2.3) \quad t^m_n \to \infty \quad \text{and} \quad \pi(t^m_n, x_n) \to y_n$$

for $m \to \infty$. 
So for every $n$ there is an $m_n$ such that putting $s_n := t_{m_n}^n$ we get

$$s_n \geq n \quad \text{and} \quad \rho(\pi(s_n, x_n), y_n) < \frac{1}{n}.$$  \hfill (2.4)

Elementary reasoning gives:

$$\rho(\pi(s_n, x), y) \leq \rho(\pi(s_n, x), \pi(s_n, x_n)) + \rho(\pi(s_n, x_n), y_n) + \rho(y, y_n).$$  \hfill (2.5)

The positive stability of $\pi^x$ allows us to make the first term of the right hand side of (2.5) as small as we wish (since $x_n \to x$), the second term tends to zero (see (2.4)), the third one also tends to zero as $n \to \infty$, which means (because of the first condition of (2.4)) that $y \in \Lambda^+(x)$.

**Remark 2.4.** Observe that we did not need the continuity of $\pi$; we may use the semi-stability of $\pi^x$ instead of stability since the first term of the right hand side of (2.5) should be small only for large $n$ (not necessarily for every $n$). The semi-stability however was indeed necessary in the proof presented above. It is easy to show that this condition is essential not only because of the method used in the proof; an example could be produced even in the case $X = \mathbb{R}^2$.

It is enough to consider a dynamical system $(\mathbb{R}^2, \mathbb{R}, \pi)$ having the trajectories presented in the picture (Fig. 1).

![Fig. 1](image)

It is clear that $\Lambda^+(x) = \{\tilde{y}\}$ while for every $z = (z_1, z_2) \in \mathbb{R}^2$ such that $|z_2| > 0$, $z_1 < 1$, $\Lambda^+(z) = \{w\}$ where $w = (w_1, w_2)$, $w_1 = 2$, $w_2 = z_2$, and so the mapping (2.1) is not (H)-usc at $x$. Observe that here $\pi^x$ is not (positively) stable. Observe that in this case the mapping

$$J^+: y \mapsto J^+(y)$$  \hfill (2.6)
as well as the mapping

\[(2.7) \quad D^+: y \mapsto D^+(y)\]

is (H)-usc at \(x\).

The above observation can be generalized: for the mappings (2.6) and (2.7), as well as for \(J^-: y \mapsto J^-(y)\), and \(D^+: y \mapsto D^+(y)\) to be (H)-usc the semi-stability of \(\pi^x\) is not needed. More precisely, we have the following

**Theorem 2.2.** The mappings \(J^+, J^-, D^+\) and \(D^-\) are (H)-usc at every point.

**Proof.** We shall limit ourselves to the case of \(J^+\) only. Let \(\{x_n\}\) and \(\{y_n\}\) be such that

\[(2.8) \quad y_n \in J^+(x_n) \quad \text{for every } n,\]

\[(2.9) \quad x_n \to x \quad \text{as } n \to \infty\]

and

\[(2.10) \quad y_n \to y \quad \text{as } n \to \infty.\]

The condition (2.8) means that for every \(n\) there are sequences \(\{t^n_m\}\) and \(\{y^n_m\}\) such that

\[(2.11) \quad t^n_m \to \infty \quad \text{as } m \to \infty\]

\[(2.12) \quad y^n_m \to x_n \quad \text{as } m \to \infty\]

and

\[(2.13) \quad \pi(t^n_m, y^n_m) \to y_n \quad \text{as } m \to \infty.\]

So, for every \(n\) there exists \(m_n\) such that for \(s_n := t^n_{m_n}\) and \(z_n := y^n_{m_n}\) we have

\[(2.14) \quad s_n \geq n\]

\[(2.15) \quad \rho(z_n, x_n) < \frac{1}{n}\]

and

\[(2.16) \quad \rho(\pi(s_n, z_n), y_n) < \frac{1}{n}.\]
From (2.15) and (2.9) we get

\[(2.17) \quad z_n \rightarrow x,\]

from (2.14) we obtain obviously

\[(2.18) \quad s_n \rightarrow \infty\]

while from (2.16) and (2.10) we get (applying a classical argument based on triangle inequality)

\[(2.19) \quad \pi(s_n, z_n) \rightarrow y.\]

The conditions (2.17) – (2.19) give finally the relation

\[y \in J^+(x)\]

which proves the assertion of our theorem for the mapping \(J^+\).

The proofs of the other cases are quite similar.

It is impossible to prove similar theorems obtained from Theorem 2.1 and 2.2 by substituting (C)-usc in the place of (H)-usc if we do not state certain additional assumptions. Even the assumption on local compactness of \(X\) is not sufficient for (C)-usc of the mappings (2.1), \(J^+\), \(D^+ (J^-, D^-)\), which is shown in the following example.

Let us consider a dynamical system in \(X = \mathbb{R}^3 \setminus \gamma\) where

\[\gamma := \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 = 0, y_2^2 + y_3^2 = 1, y_2 \geq 0\}.\]

Every point \(y = (y_1, y_2, y_3) \in X\) being of the form \((y_1, 0, 0)\) as well as every point \(z = (z_1, z_2, z_3)\) such that \(z_2^2 + z_3^2 \geq 1\) is a stationary point. For every \(\vec{x} = (\vec{x}_1, \vec{x}_2, \vec{x}_3) \in X\) such that \(\vec{x}_2^2 + \vec{x}_3^2 < 1\), the trajectory lies in the plane \(P(\vec{x}_1) = \{(x_1, x_2, x_3) : x_1 = \vec{x}_1\}\) and is a spiral curve approaching the unit circle in \(P(\vec{x}_1)\) if the time tends to infinity and approaching the origin of that circle for the time going to minus infinity; more precisely: for every \(\vec{x}\) such that \(\vec{x}_2^2 + \vec{x}_3^2 < 1\), we have

\[\Lambda^+ (\vec{x}) = \left\{(\vec{x}_1, w_2, w_3) : w_2^2 + w_3^2 = 1\right\} \setminus \gamma\]
and $\Lambda^-(\tilde{x}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ x_1, 0, 0 \end{pmatrix} \right\}$. Assume that $\pi^x$ is a positively stable motion for every $x = (x_1, x_2, x_3)$ such that $0 < x_2^2 + x_3^2 < 1$. Consider now for instance $x^* = (0, \frac{1}{2}, \frac{1}{2})$. We have

$$\Lambda^+(x^*) = \{ (0, w_2, w_3) : w_2^2 + w_3^2 = 1, w_2 < 0 \}$$

while for every $x = (x_1, x_2, x_3), x_2^2 + x_3^2 < 1, x_1 \neq 0$, we have

$$\Lambda^+(x^*) = \{ (x_1, w_2, w_3) : w_2^2 + w_3^2 = 1 \}$$

and so the mapping (2.1) is not (C)-usc at the point $x^*$. The condition (H)-usc is satisfied according to Theorem 2.1; it could be however verified directly. Indeed, there are two possibilities for sequences $\{x_n\}, \{y_n\}$ such that $x_n \to x^*, y_n \in \Lambda^+(x_n)$: either some subsequence $\{y_{k_n}\}$ of $\{y_n\}$ is convergent in $X$ to a point $y^*$ and then this limit $y^*$ belongs to $\Lambda^+(x^*)$ or none of subsequences of $\{y_n\}$ is convergent in $X$. In this second case the sequences $\{y_n\}$ considered as a sequence of elements of $\mathbb{R}^3$ may have of course a subsequence convergent to a limit belonging to $\gamma$; this curve however is removed from the space and so such a limit does not exists in $X$. The formal implication introduced in the definition of (H)-usc is in that case satisfied.

At the same time the absence of points belonging to $\gamma$ makes that the condition (C)-usc is not fulfilled.

Also, it is not difficult to show that neither $J^+$ nor $D^+$ is (C)-usc in that case, So, even the positive stability of $\pi^x$ is not sufficient for $J^+$ and $D^+$ to be (C)-usc in the space $X = \mathbb{R}^3 \setminus \gamma$.

Observe that the above space $X$ is locally compact but the set $\Lambda^+(x^*)$ is not compact.

We can modify the above example replacing $\gamma$ by

$$\tilde{\gamma} := \gamma \setminus \{(0, 0, -1), (0, 0, 1)\},$$

and considering a new dynamical system $\left( \tilde{X}, \mathbb{R}; \tilde{\pi} \right)$ such that $\tilde{X} = \mathbb{R}^3 \setminus \tilde{\gamma}$, $\tilde{\pi}(t, x) = \pi(t, x)$ for $\tilde{\pi}(t, (0, 0, -1)) = (0, 0, -1), \tilde{\pi}(t, (0, 0, 1)) = (0, 0, 1)$.

It is clear that also in this system the same phenomena for $\Lambda^+, J^+$ and $D^+$ can be observed: they are (H)-usc at $x^*$ but not (C)-usc. In this case the set $\Lambda^+(x^*)$ is compact (as well as $J^+(x^*)$ and $D^+(x^*)$) but the space $X$ is not locally compact.

The above examples show that trying to extend Theorems 2.1 and 2.2 in order to obtain similar results for (C)-usc one has to add some supplementary assumptions. This could be done for instance by considering only classical
Euclidean spaces $\mathbb{R}^k$. Another situation was considered in [7] where $\overline{\pi(x)}$ were assumed to be compact for some or for all $x$.

We will formulate precisely corresponding theorems in the next section.

**Remark 2.5.** The first statement (i) of Proposition 1.1 and Remark 3.1 on the implication (C)-usr $\implies$ (H)-usc (if $F(x)$ is closed) imply that if the mapping $y \mapsto \Lambda^+(y)$ (or $y \mapsto J^+(y)$ or $y \mapsto D^+(y)$) is (C)-usc, then it must be necessarily also (H)-usc. So it is natural to expect that conditions sufficient for (C)-usc will be stronger (at least: not weaker) than those sufficient for (H)-usc.

**3. Theorem 3.1.** Let $(\mathbb{R}^k, \mathbb{R}, \pi)$ be a dynamical system. Assume that $x \in \mathbb{R}^k$ is such that $\pi^x$ is positively (negatively) stable. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

\begin{equation}
\rho(x, y) < \delta \implies J^+(y) \subset B(\Lambda^+(x), \varepsilon) \quad (J^-(y) \subset B(\Lambda^-(x), \varepsilon)).
\end{equation}

Here $\rho$ denotes for instance the Euclidean metric in $\mathbb{R}^k$:

$$\rho(x, y) = ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2};$$

one can consider of course any other equivalent metric.

**Proof.** Observe that if $\Lambda^+(x) = \emptyset$ ($\Lambda^-(x) = \emptyset$) then $\Lambda^+(y) = J^+(y) = \emptyset$ ($\Lambda^-(y) = J^-(y) = \emptyset$) for $y$ sufficiently close to $x$ and then the condition (3.1) is obviously satisfied. So we may assume that $\Lambda^+(x) \neq \emptyset$ ($\Lambda^-(x) \neq \emptyset$).

Assume that the motion $\pi^x$ is positively stable. Let $\varepsilon > 0$ be given. Choose a $\delta^0 > 0$ in such a way that

\begin{equation}
\rho(x, y) < \delta^0 \implies \rho(\pi(t, x), \pi(t, y)) < \frac{\varepsilon}{2} \quad \text{for} \quad t \geq 0;
\end{equation}

such a choice is possible since $\pi^x$ is positively stable.

We shall show the condition (3.1) is satisfied for $\delta = \delta^0/2$ (with respect to $J^+$, of course).

Suppose the contrary. So there is a point, say $y^0$, belonging to $B(x, \frac{\delta^0}{2})$ such that

\begin{equation}
J^+(y^0) \setminus B(\Lambda^+(x), \varepsilon) \neq \emptyset.
\end{equation}

Let $y^*$ be a point of the set $J^+(y^0)$ such that

\begin{equation}
y^* \notin B(\Lambda^+(x), \varepsilon).
\end{equation}
Let \( \{t_n\} \) and \( \{y_n\} \) be sequences of real numbers and \(-\) respectively \(-\) elements of \( \mathbb{R}^k \) such that

\[
\begin{align*}
(3.5) \quad t_n & \to \infty \\
(3.6) \quad y_n & \to y^0 \\
(3.7) \quad \pi(t_n, y_n) & \to y^*
\end{align*}
\]

as \( n \to \infty \).

For \( n \) sufficiently large we have (compare (3.6))

\[
\rho(y_n, y^0) < \frac{\delta^0}{2}
\]

and so, by virtue of the fact that \( \rho(y^0, x) < \frac{\delta^0}{2} \), we get from (3.2)

\[
(3.8) \quad \rho(\pi(t_n, y_n), \pi(t_n, x)) < \frac{\varepsilon}{2}
\]

for \( n \) large enough.

The conditions (3.4) and (3.7) imply

\[
(3.9) \quad \rho(\pi(t_n, y_n), \Lambda^+(x)) > \frac{\varepsilon}{2}
\]

for large \( n \).

The conditions (3.7) and (3.8) (satisfied for sufficiently large \( n \)) imply that the sequence \( \{\pi(t_n, x)\} \) is bounded. So we may assume, without loss of generality, that this sequence is convergent to some element \( z^* \in \mathbb{R}^k \). It is clear that \( z^* \in \Lambda^+(x) \). Using (3.8) we get

\[
\rho(y^*, z^*) \leq \frac{\varepsilon}{2} < \varepsilon
\]

and so \( y^* \in B(\Lambda^+(x), \varepsilon) \) which contradicts directly (3.4). The proof for \( J^+(y) \) is finished.

The proof in the second case is quite similar.

Remark 3.1. The continuity of \( \pi \) was not needed in the proof of Theorem 3.1; so it is true for pseudo-dynamical systems as well as with the more general assumption on \( \pi^x \), namely with the semi-stability conditions instead of stability ones.
COROLLARY 3.1. If \((R^k, R; \pi)\) is a dynamical system and \(x \in R^k\) is such that \(\pi^x\) is positively (negatively) stable, then the mapping

\[
\Lambda^+: y \mapsto \Lambda^+(y) \quad (\Lambda^-: y \mapsto \Lambda^-(y))
\]

is \((C)\)-usc at the point \(x\), as well as the mapping \(J^+ (J^-)\).

In order to prove this statement with respect to \(\Lambda^+ (\Lambda^-)\) we observe that for every \(z\) we have

\[(3.10) \quad \Lambda^+(z) \subset J^+(z) \quad (\Lambda^- (z) \subset J^-(z)),\]

and apply Theorem 3.1. The condition (3.10) implies clearly the inclusion

\[
B(\Lambda^+(x), \varepsilon) \subset B(J^+(x), \varepsilon) \quad (B(\Lambda^-(x), \varepsilon) \subset B(J^-(x), \varepsilon))
\]

which gives the assertion for \(J^+ (J^-)\) again by virtue of Theorem 3.1.

COROLLARY 3.2. If \((R^k, R; \pi)\) is a dynamical system and \(x \in R^k\) is such that \(\pi^x\) is positively (negatively) stable, then the mapping \(D^+ (D^-)\) is \((C)\)-usc at \(x\).

PROOF. It is known that in dynamical systems the positive (negative) stability of motions implies the positive (negative) stability of trajectories. So, in particular, if \(\pi^x\) is positively (negatively) stable then for every \(\varepsilon > 0\) there is a \(\delta > 0\) such that

\[(3.11) \quad \rho(x, y) < \delta \implies \pi^+(y) \subset B(\pi^+(x), \varepsilon) \quad (\pi^- (y) \subset B(\pi^-(x), \varepsilon)).\]

Thus it is enough to recall (iii) from Proposition 1.1 and apply Theorem 3.1.

COROLLARY 3.3. If \((R^k, R; \pi)\) is a dynamical system and \(x \in R^k\) is such that \(\pi^x\) is positively (negatively) stable, then the mapping

\[
\overline{\pi}^+: y \mapsto \overline{\pi}^+(y) \quad (\overline{\pi}^+: y \mapsto \overline{\pi}^-(y))
\]

is \((C)\)-usc at \(x\).

PROOF. We apply (ii) from Proposition 1.1, the stability condition mentioned in the proof of Corollary 3.2 (see (3.11)) and Theorem 3.1.
4. Examples. Recall that for (H)-usc at $x$ of $J^+$ and $D^+$ we do not need the positive stability of $\pi^x$ (see Theorem 2.2). Corollaries 3.1 and 3.2 of Theorem 3.1 show that the positive stability of $\pi^x$ is sufficient for $J^+$ and $D^+$ to be (C)-usc at $x$. Moreover, stability of $\pi^x$ was essentially needed in the proof because of the method applied there. It is now very natural to state the question: is this stability really essential or was it necessary only for the method used above? The answer is given by the following examples of dynamical system $\left(\mathbb{R}^2, \mathbb{R}; \pi\right)$ which has trajectories presented on the picture (Fig. 2)

![Diagram of dynamical system](image)

**Fig. 2**

Here

\[
\Lambda^+(x^*) = \{y\}, \\
J^+(x^*) = \{(z_1, y_2) : z_1 \geq y_1\}, \\
D^+(x^*) = \pi_+(x^*) \cup J^+(x^*) = \{(z_1, x_2^*), z_1 \geq x_1^*\}
\]

and it is clear that none of the mappings $\Lambda^+, J^+, D^+$ is (C)-usc at $x^*$. Observe that the motion $\pi^{x^*}$ is not (positively) stable. Moreover, it is obvious that the trajectory $\pi(x^*)$ is not (positively) stable, as well as the positive semitrajectory $\pi_+(x^*)$. So, the next natural question arises: is it possible to construct such a dynamical system in which the mappings $\Lambda^+, J^+$ and $D^+$ will be not (C)-usc at a given point $x^0$, while the trajectory $\pi(x^0)$ will be positively stable as a subset of $X$ (cf. [1], [2]), which means in our case (because of the continuity of $\pi$) that: for every $\varepsilon > 0$ there exists $\delta > 0$ such that the condition (3.11) holds for $x = x_0$. A positive answer to that question is given by the following example. Let $\left(\mathbb{R}^2, \mathbb{R}; \bar{\pi}\right)$ be a dynamical having trajectories presented in the picture (Fig. 3).
The curves $K_1, K_2, K_3, \ldots$ have as their elements stationary points. The curve $K_n$ lies between straight lines passing through points $(0, 2^{-n})$ and $(0, 2^{-n-1})$ parallel to the first axis of coordinates. Exactly one point of $K_n$ ("the top of $K_n"$) has the coordinates $(n, 2^{-n})$; $K_n$ approaches asymptotically the straight line $\{(x, 2^{-n-1}) : x \in \mathbb{R}\}$ as it is presented in the picture. For every point $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_2 > 0$, the trajectory $\tilde{\pi}(x)$ is such that either $\Lambda^+(x) = \{(n, x_2)\}$ for some positive integer $n$ or $\Lambda^+(x) = \emptyset$, and in this case $\tilde{\pi}_+(x)$ considered as a curve in the plane $\mathbb{R}^2$ is the union of some segment (parallel to the first axis) and a parabol-like curve (see the picture). For every $x \in \mathbb{R}^2$ such that $x_2 \leq 0$, the trajectory $\tilde{\pi}(x)$ is a straight line parallel to the first coordinate axis.

It is not difficult to observe that $J^+((0, 0)) = \emptyset$ and

$$D^+((0, 0)) = \tilde{\pi}_+((0, 0)) = \{(x, 0) : x \geq 0\}.$$ 

The mapping $J^+$ and $D^+$ are not (C)-usc at $(0, 0)$ but they are (H)-usc. The motion $\tilde{\pi}^{(0,0)}$ is not positively stable, but the trajectory $\pi((0, 0))$ (as well as the semi-trajectory $\tilde{\pi}_+((0, 0))$) is positively stable.

Let us modify the above example. Instead of parabola-like segments of curves, consider straight half-lines. So all trajectories being not stationary
points (placed still on the curves $K_1, K_2, \ldots$) are now straight lines or half-lines parallel to the $x_1$-axis. In that case the positive semi-trajectory of the point $(0,0)$ is not only positively stable, but even positively uniformly stable. The mappings $\Lambda^+$ and $J^+$ are not (C)-usc.

References


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