HOMOGENIZATION
OF HYPERBOLIC–PARABOLIC EQUATIONS
IN PERFORATED DOMAINS

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1. Introduction. In this note we examine the following equation

\[ \alpha_\varepsilon u''_\varepsilon + \beta_\varepsilon u'_\varepsilon + A_\varepsilon u_\varepsilon = f_\varepsilon \quad \text{in} \quad \Omega_\varepsilon \times (0,T) \]

with boundary and initial conditions

\[
\begin{align*}
\frac{\partial u_\varepsilon}{\partial \nu_{A_\varepsilon}} &= 0 & \text{on} & \quad \partial \Omega \times (0,T) \\
\frac{\partial u_\varepsilon}{\partial \nu_{\partial S_\varepsilon}} &= 0 & \text{on} & \quad \partial S_\varepsilon \times (0,T) \\
u_\varepsilon(0) &= g_\varepsilon & \text{in} & \quad \Omega_\varepsilon \\
\alpha_\varepsilon u'_\varepsilon(0) &= \sqrt{\alpha_\varepsilon} \psi_\varepsilon & \text{in} & \quad \Omega_\varepsilon.
\end{align*}
\]

Above, by \( \Omega_\varepsilon \) we denote a perforated domain obtained from a fixed domain \( \Omega \) by removing a number \( N_\varepsilon \) of holes of size \( \varepsilon \) periodically distributed, \( A_\varepsilon \) denotes a uniformly elliptic, second order differential operator of the form

\[ A_\varepsilon = -\frac{\partial}{\partial x_i} \left( a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} \right). \]

The conormal derivative on \( \partial S_\varepsilon \times (0,T) \) is given by

\[ \frac{\partial}{\partial \nu_{A_\varepsilon}} = a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} n_i, \]

where \( \partial S_\varepsilon \) denotes the boundary of the holes and the normal \( n \) is directed towards the exterior of \( \Omega_\varepsilon \). In the paper a summation convention over repeated subscripts is adopted. We want to study the behaviour of \( u_\varepsilon \) when \( \varepsilon \to 0 \).
In general, Equation of the form (1.1) (called hyperbolic-parabolic (HP-) equation [3]) models the wave processes; for $\alpha_\varepsilon = 0$, $\beta_\varepsilon \neq 0$ it appears in many problems of heat theory, for $\alpha_\varepsilon \neq 0$, $\beta_\varepsilon \neq 0$ this equation includes the phenomena studied in theories of sound, light, electricity, magnetism, in hydrodynamics and in elasticity theory (see [7], [10]).

One assume that $\alpha_\varepsilon$ and $\beta_\varepsilon$ are functions of the space variable. This situation appears very often in applications (see [1]). For example, in the study of small vibrations of the rod, the coefficient $\alpha_\varepsilon$ is a linear density of a rod, $\beta_\varepsilon$ is a dissipation coefficient, the matrix $a_{ij}$ describes a physical property of the material, $f_\varepsilon$ is the distribution of density of exterior forces. The component $\beta_\varepsilon u'_\varepsilon$ corresponds to the friction force in the case of vibration of the rod in viscous medium. Introducing this term makes easy passage from the general case of equation (1.1) to the heat equation i.e. when $\alpha_\varepsilon = 0$.

In this paper we prove a theorem on asymptotic behaviour of solutions of problem (1.1), (1.2), when the size of holes tends to zero ($\varepsilon \to 0$). We study this equation in the periodic case and we assume that the size of the holes is of the same order as the distance between adjacent holes. In the limit, the equation (1.1) is replaced by a homogenized one with constant coefficients given on the whole $\Omega$. The result of this paper extends an earlier one by Bakhvalov and Panasenko [1].

In particular, in a fixed domain, equation (1.1) was considered in [2] in two cases: $\alpha_\varepsilon = 1$, $\beta_\varepsilon = 0$ and $\alpha_\varepsilon = 0$, $\beta_\varepsilon = 1$, and in [3] for arbitrary $\alpha_\varepsilon, \beta_\varepsilon$. In a perforated domain it was studied in [4] but only in the case where $\alpha_\varepsilon = 1$, $\beta_\varepsilon = 0$, while here we consider the general case where $\alpha_\varepsilon$ is arbitrary in $L^\infty(\Omega)$ and $\beta_\varepsilon \geq c > 0$.

As regards the right hand side of the equation and the initial functions, they are often of the form $f_\varepsilon(x, t) = f(\frac{x}{\varepsilon}, t)$, $g_\varepsilon(x) = g(\frac{x}{\varepsilon})$, $\psi_\varepsilon(x) = \psi(\frac{x}{\varepsilon})$, i.e., they practically depend on a rapidly oscillating variable, (compare [1]).

2. Formulating of the problem. Let $Y = \prod_{i=1}^n [0, l_i]$ be a representative cell in $\mathbb{R}^n$ ($l_i > 0$, $n = 2, 3, \ldots$) and let $S$ be an open subset of $Y$, with a smooth boundary $\partial S$, such that $\overline{S} \subset Y$. Let $\varepsilon$ be a positive real number. For each $\varepsilon$, and for any integer vector $k$ in $\mathbb{Z}^n$, we shall denote by $T(\varepsilon, k)$ the translated image of $\varepsilon \overline{S}$ by the vector $k$, i.e.

$$T(\varepsilon, k) = \varepsilon(k + \overline{S}).$$

So, the whole $\mathbb{R}^n$ is perforated periodically by holes of $\varepsilon$-size.
Let $\Omega$ be an open, bounded subset of $\mathbb{R}^n$. We assume that the holes $T(\varepsilon, k)$ do not intersect the boundary $\partial \Omega$, which is supposed to be sufficiently regular. Let us denote by $\Omega_\varepsilon$ the perforated domain

$$\Omega_\varepsilon = \Omega \cap (\mathbb{R}^n \setminus T(\varepsilon, k)).$$

Let us observe that $\Omega_\varepsilon$ represents the subregion of $\Omega$ obtained from $\Omega$ by removing a finite number of the holes. All of them have the same shape $\varepsilon \mathcal{S}$, and they are periodically distributed in $\Omega$, with period $\varepsilon$ in each axis-direction. Consequently, we have

$$\partial \Omega_\varepsilon = \partial \Omega \cup \partial \mathcal{S}_\varepsilon,$$

where $\partial \mathcal{S}_\varepsilon$ is the subset of $T(\varepsilon, k)$ consisting of the boundaries of the holes. Notice that $Q_\varepsilon = \Omega_\varepsilon \times (0, T)$ can be regarded as a domain with cylindrical holes.

The region $\Omega_\varepsilon$ in the two-dimensional case
We use the following notations:

\[ Y^* = Y \setminus \mathcal{S}, \]
\[ |\omega| = \text{measure of the set } \omega \text{ of } \mathbb{R}^n, \]
\[ \theta = \frac{|Y^*|}{|Y|}, \]
\[ \mathcal{M}_\omega(\varphi) = \frac{1}{|\omega|} \int_\omega \varphi(s)ds; \text{ the mean value of } \varphi \text{ on } \omega, \]
\[ 0 < T < +\infty, \quad Q = \Omega \times (0, T), \]
\[ H_\epsilon = L^2(\Omega_\epsilon). \]

Let us denote by \( \chi_\omega \) the characteristic function of the set \( \omega \) and by the \( \tilde{\cdot} \) the zero extension operator to the whole \( \Omega \), i.e., for a given function \( u \) defined on \( \Omega_\epsilon \) we put:

\[
\tilde{u} = \begin{cases} 
    u & \text{on } \Omega_\epsilon \\
    0 & \text{on } \mathcal{S}_\epsilon.
\end{cases}
\]

(2.1)

Denote by \( w - X, s - X \) the space \( X \) endowed with its weak and strong topology, resp. In order to give the variational formulation of the problem, let us introduce the following space:

\[ V_\epsilon = \{ u \in H^1(\Omega_\epsilon) \mid u = 0 \text{ on } \partial \Omega \} \]

equipped with the norm

\[ \|u\|_{V_\epsilon} = \|\nabla u\|_{(L^2(\Omega_\epsilon))^n} \]

which is equivalent to the \( H^1(\Omega_\epsilon) \)-norm.

Consider now the problem (1.1), (1.2) where the coefficients \( a_{ij} \) in (1.3) satisfy the following hypothesis:

\[
\begin{cases} 
    a_{ij} = a_{ji} \in L^\infty(\mathbb{R}^n), \\
    \exists \lambda > 0 \text{ s.t. } \lambda \xi_i \xi_i \leq a_{ij}(y)\xi_i \xi_j, \quad \forall \xi \in \mathbb{R}^n, \\
    a_{ij} \text{ are } Y\text{-periodic (i.e. } a_{ij} \text{ admits period } l_k \text{ in the direction } y_k \text{ for } k = 1, \ldots, n).)
\end{cases}
\]

(2.2)

Assume that the data satisfy hypotheses:

\[
\begin{cases} 
    \alpha_\epsilon, \beta_\epsilon \in L^\infty(\Omega), \\
    0 \leq \alpha_\epsilon \text{ a.e. in } \Omega, \\
    0 < c \leq \beta_\epsilon \text{ a.e. in } \Omega,
\end{cases}
\]

(2.3)
\[
\begin{aligned}
\left\{
\begin{array}{l}
f_\varepsilon \in L^2(Q_\varepsilon), \\
g_\varepsilon \in H^1_0(\Omega), \\
\psi_\varepsilon \in H_\varepsilon.
\end{array}
\right.
\end{aligned}
\]  

(2.4)

3. **An extension result.** We begin by pointing out that the solutions of (1.1), (1.2) are defined in \(Q_\varepsilon\) only and not in the whole \(Q\), as it should be desired for the study of their asymptotic behaviour. We shall therefore introduce a family of linear extension operators.

**Lemma 3.1.** There exists an extension operator

\[ P_\varepsilon \in \mathcal{L}(L^2(0,T;H^1(\Omega_\varepsilon)),L^2(0,T;H^1(\Omega))) \cap \mathcal{L}(L^2(Q_\varepsilon),L^2(Q)) \]

such that:

(i) \( P_\varepsilon u = u \) on \( Q_\varepsilon \) for each \( u \) defined on \( \Omega_\varepsilon \times (0,T) \),

(ii) \( P_\varepsilon u' = (P_\varepsilon u)' \) in \( Q \),

(iii) \( \|P_\varepsilon u\|_{L^2(Q)} \leq k \|u\|_{L^2(Q_\varepsilon)} \) for each \( u \in L^2(Q_\varepsilon) \),

(iv) \( \|P_\varepsilon u'\|_{L^2(Q)} \leq k \|u\|_{L^2(\Omega_\varepsilon)} \) for each \( u \in L^2(0,T;H^1(\Omega_\varepsilon)) \),

(v) \( \|\nabla(P_\varepsilon u)\|_{L^2(0,T;L^2(\Omega)^n)} \leq k \|\nabla u\|_{L^2(0,T;L^2(\Omega_\varepsilon)^n)} \),

where \( k \) is a constant independent of \( \varepsilon \).

**Proof.** This proof is a slight modification of the one given in paper [4]. In order to construct an extension operator from \( Q_\varepsilon \) to \( Q \), it suffices to know how to extend functions defined on \( Y^* \times (0,T) \) to the functions given on the representative cell \( Y \times (0,T) \).

We know ([6]) that there exists at least one extension operator

\[ R \in \mathcal{L}(H^s(Y^*),H^s(Y)), \quad s = 0,1 \]

such that

\[ \|\nabla R\varphi\|_{(L^2(Y))^n} \leq c \|\nabla \varphi\|_{(L^2(Y^*))^n}, \quad \forall \varphi \in H^1(Y^*). \]

(3.1)

Indeed, for \( \varphi \in H^1(Y^*) \), in \( S \), we can take \( R\varphi \) equal to the solution of the Dirichlet problem:

\[
\begin{aligned}
\Delta v &= 0 \quad \text{in } S \\
v &= \varphi \quad \text{on } \partial S.
\end{aligned}
\]

Hence, it is clear (\( R\varphi \) being \( \varphi \) in \( Y^* \)) that the operator \( P \) defined by

\[ (Pz)(x,t) = [Rz(\cdot,t)](x), \quad \forall z \in H^1(Y^* \times (0,T)) \]
satisfies

\[ P \in \mathcal{L}(L^2(0,T; H^1(Y^*))^n, L^2(0,T; H^1(Y))) \]

Moreover, since the function \( t \mapsto z(\cdot, t) \) belongs to \( L^2(0,T; L^2(Y^*)) \) for \( z \in H^1(Y^* \times (0,T)) \), we know that the function \( t \mapsto Rz(\cdot, t) \) belongs to \( L^2(0,T; L^2(Y)) \). Notice that \( R \) is independent of \( t \) and

\[ R(z(\cdot, t)) = (Rz(\cdot, t))'. \]

Using (3.1) we can easily show that

\[ \|\nabla(Pz)\|_{L^2(0,T; (L^2(Y))^n)} \leq k\|\nabla z\|_{L^2(0,T; (L^2(Y^*))^n)}. \]

The operator \( P \) satisfies the properties (i)–(v) on the representative cell. By the transformation \( x = \varepsilon y \) we obtain the passage from \( Y \) to \( \Omega \). The construction of \( P_\varepsilon \) is the same as in Lemma 2.1 in [4] or Lemma 3 in [5].

The next lemma characterizes the weak limits of the sequences of \( \varepsilon \)-periodic functions.

**Lemma 3.2.** Let \( f \in L^2(Y) \). If we extend it periodically to \( \mathbb{R}^n \), we have

\[ f\left(\frac{x}{\varepsilon}\right) \rightharpoonup \mathcal{M}_Y(f) \quad w \in L^2(\Omega), \quad \text{as } \varepsilon \to 0. \]

The proof of this fact can be found, for instance, in [7], Ch. 5.

**4. The main result.** For problem (1.1), (1.2) we have the following asymptotic-type result:

**Theorem 4.1.** Assume that (2.2), (2.3), (2.4) hold. Let us suppose that \( \alpha_\varepsilon > 0 \) and

\begin{align*}
(4.1) \quad \alpha_\varepsilon & \rightarrow \alpha \quad s - L^\infty(\Omega) \text{ with } \alpha \geq c > 0 \text{ a.e. in } \Omega, \\
(4.2) \quad \sqrt{\alpha_\varepsilon} & \rightarrow \gamma \quad w - * - L^\infty(\Omega), \\
(4.3) \quad \beta_\varepsilon & \rightarrow \beta \quad s - L^\infty(\Omega), \\
(4.4) \quad \tilde{f}_\varepsilon & \rightarrow f \quad w - L^1(Q), \\
(4.5) \quad g_\varepsilon & \rightarrow g \quad w - H^1_0(\Omega), \\
(4.6) \quad \tilde{\psi}_\varepsilon & \rightarrow \psi \quad s - L^2(\Omega).
\end{align*}

Then, denoting by \( P_\varepsilon \) an extension operator given by Lemma 3.1, we have:

\begin{align*}
(4.7) \quad P_\varepsilon u_\varepsilon & \rightarrow u \quad w - L^2(0,T; H^1_0(\Omega)) \text{ and } s - L^2(Q), \\
(4.8) \quad P_\varepsilon u'_\varepsilon & \rightarrow u' \quad w - L^2(Q),
\end{align*}
where \( u \) is the solution of the homogenized equation

\[
\begin{align*}
\begin{cases}
\theta \alpha u'' + \theta \beta u' + A u = f \text{ in } Q \\
u = 0 \text{ on } \partial \Omega \times (0, T) \\
u(0) = \frac{1}{\theta} g \text{ in } \Omega \\
u'(0) = \frac{\gamma}{\theta \alpha} \psi \text{ in } \Omega.
\end{cases}
\end{align*}
\]

(4.9)

Moreover, the homogenized operator \( A = -q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \) has constant coefficients

\[
q_{ij} = \frac{1}{|Y|} \int_{Y^*} \left( a_{ij} - a_{kj} \frac{\partial \mu^i}{\partial y_k} \right) dy,
\]

(4.10)

where the functions \( \mu^i \) are solutions of

\[
\begin{align*}
\begin{cases}
-\frac{\partial}{\partial y_l} \left( a_{kl} \frac{\partial (\mu^i - y_i)}{\partial y_k} \right) = 0 \text{ in } Y^*, \\
a_{kl} \frac{\partial (\mu^i - y_i)}{\partial y_k} n_l = 0 \text{ on } \partial S, \\
\mu^i \text{ are } Y\text{-periodic}.
\end{cases}
\end{align*}
\]

(4.11)

**Remark 4.2.** We do not restrict the generality assuming the condition (4.2) together with (4.1) since having (4.1), one can always take a subsequence which satisfies (4.2). Of course, in general \( \gamma \neq \sqrt{\alpha} \). If \( \alpha_\varepsilon = 0 \) a.e. in \( \Omega \) then the equation (1.1) as well as, the limit equation becomes of the parabolic type.

**Remark 4.3.** One should underline (following [3]) that the initial condition \( \alpha_\varepsilon u'_\varepsilon(0) = \sqrt{\alpha_\varepsilon} \psi_\varepsilon \) gives a condition on \( u'_\varepsilon(0) \) only on a set where \( \alpha_\varepsilon \neq 0 \). In the homogenized (limit) equation, there is some sort of increase in the initial data, because in (4.9) the condition on \( u'(0) \) is given over the whole domain \( \Omega \).

**Proof of Theorem 4.1.** We break the proof up into a series of steps.

**Step 1 (A priori estimates).**
Our starting point for the study of problem (1.1), (1.2) is to give the "a priori" estimates of the solutions. The assumptions on the data assure the existence and the uniqueness of a solution \( u_\varepsilon \) of our problem (compare Theorem 1.1 in [3]) such that:

\[
\begin{align*}
u_\varepsilon & \in L^\infty(0, T; V_\varepsilon), \\
u'_\varepsilon & \in L^2(0, T; H_\varepsilon), \quad \sqrt{\alpha_\varepsilon} u'_\varepsilon \in L^\infty(0, T; H_\varepsilon), \\
\alpha_\varepsilon u''_\varepsilon & \in L^2(0, T; V'_\varepsilon).
\end{align*}
\]
Multiplying (1.1) by $u'_\varepsilon$ one obtains:

$$\frac{1}{2} \frac{\partial}{\partial t} [\langle \alpha_\varepsilon u'_\varepsilon, u'_\varepsilon \rangle + \langle A_\varepsilon u_\varepsilon, u_\varepsilon \rangle] + \langle \beta_\varepsilon u'_\varepsilon, u'_\varepsilon \rangle = \langle f_\varepsilon, u'_\varepsilon \rangle.$$  

Hence, integrating both sides, we get

$$\langle \alpha_\varepsilon u'_\varepsilon(t), u'_\varepsilon(t) \rangle + \langle A_\varepsilon u_\varepsilon(t), u_\varepsilon(t) \rangle + 2 \int_0^t \langle \beta_\varepsilon u'_\varepsilon, u'_\varepsilon \rangle \, ds \leq \int_0^t \langle f_\varepsilon, u'_\varepsilon \rangle \, ds + \| \psi_\varepsilon \|^2_{H_\varepsilon} + \langle A_\varepsilon g_\varepsilon, g_\varepsilon \rangle.$$  

Now, using assumptions (2.2), (2.3) we obtain (for $t \in [0,T]$):

$$\lambda \| u_\varepsilon(t) \|^2_{V_\varepsilon} + \| \sqrt{\alpha_\varepsilon} u'_\varepsilon(t) \|^2_{H_\varepsilon} + 2c \int_0^t \| u'_\varepsilon(s) \|^2_{H_\varepsilon} \, ds \leq \int_0^t \| f_\varepsilon(s) \|^2_{H_\varepsilon} \, ds + \int_0^t \| u'_\varepsilon(s) \|^2_{H_\varepsilon} \, ds + \| \psi_\varepsilon \|^2_{H_\varepsilon} + c \| g_\varepsilon \|^2_{V_\varepsilon},$$

so that

$$\| u_\varepsilon(t) \|^2_{V_\varepsilon} + \| \sqrt{\alpha_\varepsilon} u'_\varepsilon(t) \|^2_{H_\varepsilon} + c \int_0^t \| u'_\varepsilon(s) \|^2_{H_\varepsilon} \, ds \leq c \left( \| f_\varepsilon \|^2_{L^2(Q_\varepsilon)} + \| \psi_\varepsilon \|^2_{H_\varepsilon} + \| g_\varepsilon \|^2_{V_\varepsilon} \right).$$

It follows from this inequality that

(4.12) $\| u_\varepsilon \|_{L^\infty(0,T;V_\varepsilon)} \leq c,$

(4.13) $\| \sqrt{\alpha_\varepsilon} u'_\varepsilon \|_{L^\infty(0,T;H_\varepsilon)} \leq c,$

(4.14) $\| u'_\varepsilon \|_{L^2(0,T;H_\varepsilon)} \leq c,$

where $c$ is a constant independent of $\varepsilon$. By using these "a priori" estimates, we can now proceed to describe the asymptotic behaviour of the solutions of problem (1.1), (1.2), as $\varepsilon \to 0$.

**STEP 2 (CONVERGENCE OF $u_\varepsilon$)**

Let $P_\varepsilon$ be an extension operator given by Lemma 3.1. Then, (4.12), (4.14) and (ii) od Lemma 3.1 imply that for a subsequence we have:

(4.15) \[ \begin{cases} P_\varepsilon u_\varepsilon \longrightarrow u & w \in L^2(0,T;H_0^1(\Omega)), \\ (P_\varepsilon u_\varepsilon)' \longrightarrow u' & w \in L^2(Q), \end{cases} \]
for some function \( u \in L^2(0,T;H^1_0(\Omega)) \). Now, in view of (4.15), using the compactness result (see [9] p.85, Corollary 4) one gets:

\[
(4.16) \quad P_\varepsilon u_\varepsilon \longrightarrow u \quad s - L^2(Q).
\]

In the sequel we also need the following convergences (the extension \( \tilde{u}_\varepsilon \) is given by (2.1)):

\[
(4.17) \quad \begin{cases} 
\alpha_\varepsilon \tilde{u}_\varepsilon^{l'} \longrightarrow \theta \alpha u' & w - \ast - L^\infty(0,T;L^2(\Omega)), \\
\beta_\varepsilon \tilde{u}_\varepsilon^{l'} \longrightarrow \theta \beta u' & w - L^2(Q).
\end{cases}
\]

We show the first convergence in (4.17). The proof of the second one is analogous, so it will be omitted. Using (4.13) and the fact that \( \{\sqrt{\alpha_\varepsilon}\} \) is bounded in \( L^\infty(\Omega) \), we conclude that \( \{\alpha_\varepsilon u_\varepsilon^{l'}\} \) belongs to a bounded set in \( L^\infty(0,T;H_\varepsilon) \). So, up to a subsequence, we can assume that

\[
(4.18) \quad \alpha_\varepsilon \tilde{u}_\varepsilon^{l'} \longrightarrow l \quad w - \ast - L^\infty(0,T;L^2(\Omega))
\]

with some \( l \in L^\infty(0,T;L^2(\Omega)) \). We show that \( l = \theta \alpha u' \). This is a consequence of the obvious equality

\[
\iint_Q \alpha_\varepsilon \tilde{u}_\varepsilon^{l'} \phi v \, dx \, dt = -\iint_Q \alpha_\varepsilon (P_\varepsilon u_\varepsilon) \chi_{\Omega_\varepsilon} \phi v' \, dx \, dt,
\]

where \( \phi \in \mathcal{D}(\Omega), \, v \in \mathcal{D}((0,T)) \), in which we pass to the limit as \( \varepsilon \to 0 \), using (4.1), (4.16), (4.18) and the fact

\[
(4.19) \quad \chi_{\Omega_\varepsilon} \longrightarrow \theta \quad w - \ast - L^\infty(\Omega).
\]

**STEP 3 (THE LIMIT EQUATION)**

For each \( j = 1, \ldots, n \) let us define

\[
(\xi_\varepsilon)_j = a_{ij} \left( \frac{x}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial x_i}.
\]

Since \( \{a_{ij}\} \) and \( \{u_\varepsilon\} \) satisfy (2.1) and (4.12), resp., we deduce that

\[
\|\xi_\varepsilon\|_{L^\infty(0,T;(L^2(\Omega_\varepsilon))^n)} \leq c
\]

and hence

\[
(4.20) \quad \tilde{\xi}_\varepsilon \longrightarrow \xi \quad w - \ast - L^\infty(0,T;(L^2(\Omega))^n)
\]
with $\xi \in L^\infty(0,T;(L^2(\Omega))^n)$. The extension $\tilde{\xi}_\varepsilon$ is defined in (2.1). Now, take $\phi \in \mathcal{D}(\Omega)$, $v \in \mathcal{D}((0,T))$, as before. Multiplying the equation (1.1) by $\phi v$, integrating the both sides over $[0,T]$ and then using the condition $\xi_\varepsilon \cdot n = 0$ on $\partial S_\varepsilon \times (0,T)$ (implied by (1.2)), as well as integrating by parts, we obtain

$$\iint_Q \alpha_\varepsilon (P_\varepsilon u_\varepsilon) \chi_{\Omega_\varepsilon} \phi v'' \, dx \, dt - \iint_Q (P_\varepsilon u_\varepsilon) \chi_{\Omega_\varepsilon} \phi v' \, dx \, dt$$

$$+ \iint_Q \tilde{\xi}_\varepsilon \nabla \phi v \, dx \, dt = \iint_Q \tilde{f}_\varepsilon \phi v \, dx \, dt.$$

Using the convergences (4.1), (4.3), (4.4), (4.16), (4.19) and (4.20) and passing to the limit in the above identity we get

$$\theta \iint_Q \alpha u \phi v'' \, dx \, dt - \theta \iint_Q \beta u \phi v' \, dx \, dt$$

$$+ \theta \iint_Q \xi \nabla \phi v \, dx \, dt = \iint_Q \phi v \, dx \, dt.$$

It is due to the fact that $\phi$ and $v$ were arbitrary the last equality implies

$$\theta \alpha u'' + \theta \beta u' - \text{div} \xi = f \text{ in } Q. \tag{4.21}$$

In order to compute $\xi$ we use the method of Tartar, as in papers [4], [5]. For any $\lambda \in \mathbb{R}^n$ let us denote by $w_\lambda$ the solution in $H^1(Y^*)$ of the problem:

$$\begin{cases} 
- \frac{\partial}{\partial y_j} \left( a_{ij}(y) \frac{\partial w_\lambda}{\partial y_i} \right) = 0 & \text{in } Y^* \\
\frac{\partial w_\lambda}{\partial \nu_A} = 0 & \text{on } \partial S \\
w_\lambda - \lambda \cdot y & \text{Y-periodic.}
\end{cases} \tag{4.22}$$

We set $w_\varepsilon(x) = \varepsilon (Rw_\lambda) \left( \frac{x}{\varepsilon} \right)$, where $R$ is the extension operator given in the proof of Lemma 3.1, still satisfying (3.1). Owing to (3.1) we obtain

$$\begin{cases} 
w_\varepsilon \rightarrow w = \langle \lambda, \cdot \rangle & w = H^1(\Omega), \\
\nabla w_\varepsilon \rightarrow \nabla w = \lambda & w = (L^2(\Omega))^n. \tag{4.23}
\end{cases}$$

Let us introduce

$$(\eta^\lambda)_j = a_{ij}(y) \frac{\partial w_\lambda}{\partial y_i}$$

and put

$$(\eta_\varepsilon^\lambda)_j(x) = (\eta^\lambda)_j \left( \frac{x}{\varepsilon} \right) \text{ for } j = 1, \ldots, n.$$
In view of (4.22), we have

\begin{equation}
- \text{div}\, \tilde{\eta}_\lambda = 0 \quad \text{in } \Omega.
\end{equation}

On the other hand, since \( w_\lambda \) is linear in \( \lambda \) and the zero extension operator is linear, we can introduce the following matrix:

\[ \mathcal{B}_\lambda = \mathcal{M}_Y (\eta^\lambda) \quad \text{for each } \lambda \in \mathbb{R}^n. \]

Using Lemma 3.2 we observe that

\begin{equation}
\tilde{\eta}_\lambda^\lambda \rightarrow \mathcal{B}_\lambda \ w - L^2(\Omega).
\end{equation}

Now, inferring as before, i.e., multiplying (1.1) by \( \nu \phi w_\epsilon \) and (4.24) by \( \nu \phi P_\epsilon u_\epsilon \), where \( \nu \in \mathcal{D}((0,T)) \), \( \phi \in \mathcal{D}(\Omega) \), integrating both sides and so on, we obtain

\[ \int \int_Q \alpha_\epsilon (P_\epsilon u_\epsilon) \chi_{\Omega_\epsilon} w_\epsilon v'' \phi \, dx \, dt - \int \int_Q \beta_\epsilon (P_\epsilon u_\epsilon) \chi_{\Omega_\epsilon} w_\epsilon \phi' \, dx \, dt \\
+ \int \int_Q \tilde{\xi}_\epsilon \nabla w_\epsilon \nu \phi \, dx \, dt + \int \int_Q \tilde{\xi}_\epsilon \nu \phi \, dx \, dt.
\]

In turn, from the definitions of \( \xi_\epsilon \) and \( \eta_\lambda^\lambda \) and from convergences (4.1), (4.3), (4.4), (4.16), (4.19), (4.20), (4.23) and (4.25) we conclude that

\[ \theta \int \int_Q \alpha u w \phi v'' \, dx \, dt - \theta \int \int_Q \beta u \phi' v \, dx \, dt + \int \int_Q \xi \phi \, dx \, dt \\
- \int \int_Q (\mathcal{B}_\lambda) u \phi \, dx \, dt = \int \int_Q f \phi \, dx \, dt.
\]

This equality together with equation (4.21) imply that

\[ \xi = \mathcal{B}^T \nabla u. \]

Now, set \( \mu^k = -w_{\lambda_k} + y_k \) where \( \lambda_k = e_k, \{ e_j \}_{j=1}^n \) is the canonical basis of \( \mathbb{R}^n \). A simple computation shows that \( \mu^k \) is the solution of (4.11) and that \( \{ \beta_{ij} \} \) is a matrix with constant coefficients given by \( \beta_{ij} = q_{ij} \), where \( q_{ij} \) are given by (4.10). Hence \( u \) is a solution of the following limit equation:

\begin{equation}
\theta \alpha u'' + \theta \beta u' - \text{div} \left( \mathcal{B}^T \nabla u \right) = f \quad \text{in } Q.
\end{equation}
STEP 4 (The limit initial conditions)

It remains to verify that $u$ satisfy two initial conditions in (4.9). They are easily obtained by choosing suitable test functions. Let us take $\phi \in D(\Omega)$ and $v \in D([0, T])$ with $v(T) = 0$. Multiplying (1.1) by $v\phi$ and then integrating by parts over $Q$ one gets:

$$
- \int_\Omega \sqrt{\alpha \psi} \tilde{\psi}_\varepsilon v(0) \phi \, dx - \int_Q \alpha \tilde{u}_\varepsilon v' \phi \, dt + \int_Q \beta \tilde{u}_\varepsilon v\phi \, dx dt
+ \int_Q \tilde{\xi} v \nabla \phi \, dx dt = \int_Q f \varepsilon v \phi \, dx dt.
$$

Hence, letting $\varepsilon \to 0$ and using (4.2), (4.4), (4.6), (4.17), (4.20), we get the following limit identity:

$$
- \int_\Omega \gamma \psi v(0) \phi \, dx - \theta \int_Q \alpha u' v' \phi \, dx dt + \theta \int_Q \beta u' v\phi \, dx dt
+ \int_Q \xi \nabla \phi v \, dx dt = \int_Q f v \phi \, dx dt.
$$

Owing to (4.21), this equality gives:

$$
\int_\Omega [\theta \alpha u'(0) - \gamma \psi] v(0) \phi \, dx = 0;
$$

hence we obtain the second initial condition in (4.9).

Using an argument similar to that above we can show that $u(0) = \frac{1}{\partial} g$ in $\Omega$. So the proof of the theorem is completed.

Now, we consider the following problem:

$$
\begin{aligned}
\alpha \varepsilon u''_\varepsilon + \beta \varepsilon u'_\varepsilon + A \varepsilon u_\varepsilon &= f_\varepsilon \quad \text{in } Q_\varepsilon, \\
u_\varepsilon &= 0 \quad \text{on } \partial \Omega \times (0, T) \quad \text{and } \partial S_{\varepsilon} \times (0, T), \\
u_\varepsilon(0) &= g_\varepsilon \quad \text{in } Q_\varepsilon, \\
\alpha \varepsilon u'_\varepsilon(0) &= \sqrt{\alpha \varepsilon} \psi_\varepsilon \quad \text{in } \Omega_\varepsilon.
\end{aligned}
$$

(4.27)

This means that we have the Dirichlet boundary condition on the boundary of the holes. This problem can be treated in a similar way as (1.1), (1.2) but with $V_\varepsilon = H^1_0(\Omega_\varepsilon)$. In this case we can formulate the following

**Theorem 4.4.** Assume that (2.2), (2.3), (2.4) hold and $u_\varepsilon$ is the solution of (4.27). Then, there exists the extension operator $P_\varepsilon$ such that

$$
P_\varepsilon u_\varepsilon \to 0 \quad w - L^2(0, T; H^1_0(\Omega)) \text{ and } s - L^2(Q),
P_\varepsilon u'_\varepsilon \to 0 \quad w - L^2(Q),
$$
and, in consequence, we get the only, trivial, solution \( u \equiv 0 \) of the limit (homogenized) problem.

**Proof.** As in Theorem 4.1, the following estimates can be obtained

\[
\|u_\varepsilon\|_{L^\infty(0,T; H^1_0(\Omega_\varepsilon))} \leq c, \\
\|u'_\varepsilon\|_{L^2(0,T; H_\varepsilon)} \leq c.
\]

Let \( (P_\varepsilon u_\varepsilon)(x,t) = [\overline{u_\varepsilon(\cdot,t)}](x) \) (the extension is defined in (2.1)). Then, we have (for a subsequence):

\[
P_\varepsilon u_\varepsilon \rightharpoonup u_0 \quad \text{in} \quad L^2(0,T; H^1_0(\Omega)) \\
P_\varepsilon u'_\varepsilon \rightharpoonup u'_0 \quad \text{in} \quad L^2(Q),
\]

with some \( u_0 \in L^2(0,T; H^1_0(\Omega)) \). By the compactness result ([9]):

\[
(4.28) \quad P_\varepsilon u_\varepsilon \longrightarrow u_0 \quad s - L^2(Q).
\]

But

\[
\iint_Q (1 - \chi_\varepsilon) P_\varepsilon u_\varepsilon \, dxdt = 0.
\]

From (4.19) and (4.28) we get in the limit, as \( \varepsilon \to 0 \), that:

\[
\iint_Q (1 - \theta) u_0 \, dxdt = 0,
\]

hence \( u_0 = 0 \).

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**References**


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