ONE-PARAMETER POSITIVE CONTRACTION SEMIGROUPS ARE CONVERGENT

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Abstract. We show that one-parameter positive contraction semigroups on Banach lattices are convergent. For classical semigroups, like stochastic ones on $L^1(m)$, or Markovian ones on $C(Z)$, the typical limit operator is a one-dimensional and strictly positive projection (i.e. it has the form $\Lambda \otimes f_*$ or, respectively, $\mu \otimes 1$, where $\Lambda(f) = \int f \, dm$, $f_* > 0$ a.e., and $\text{supp}(\mu) = Z$).

Let $E$ be a Banach lattice with a fixed norm $\| \cdot \|$. A family $\{T(t)\}_{t \geq 0}$ of linear and bounded operators on $E$ is said to be a semigroup if $T(0) = \text{Id}$ (identity operator), $T(t_1 + t_2) = T(t_1)T(t_2)$ for all $t_1, t_2 \geq 0$, and $\lim_{t \to 0^+} \| T(t)x - x \| = 0$ for each $x \in E$. If all operators $T(t)$ are positive then the semigroup $\mathcal{T} = \{T(t)\}_{t \geq 0}$ is called positive. In this paper we deal only with positive contraction semigroups (i.e. for all $t \geq 0$ we have $\| T(t) \| \leq 1$, where $\| \cdot \|$ stands for the operator norm in $\mathcal{L}(E)$, the Banach algebra of bounded linear operators on $E$).

For a fixed operator $\rho$ – norm – closed set $\mathcal{C} \subseteq \mathcal{L}(E)$ of contractions we define $\mathcal{M}_C$ to be the family of those semigroups $\mathcal{T}$ for which $T(t) \in \mathcal{C}$ for all $t \geq 0$. Hence, we always assume that the identity operator $\text{Id}$ is in $\mathcal{C}$. Let us equip $\mathcal{M}_C$ with the metric $\rho(\mathcal{T}_1, \mathcal{T}_2) = \sup_{t \in [0,1]} \| T_1(t) - T_2(t) \|$. Then $(\mathcal{M}_C, \rho)$ becomes a complete metric space. Moreover, it is easy to calculate that for any $t_0$ we have $\sup_{t \in [0,t_0]} \| T_1(t) - T_2(t) \| \leq (1+t_0)\rho(\mathcal{T}_1, \mathcal{T}_2)$. This implies that the metric considered by Lasota and Myjak in [LaMy] (also denoted by $\rho$) is equivalent to the defined here metric $\rho$. 
In [LaMy] the authors study semigroups of stochastic operators on $L^1(m)$, where $m$ is a fixed $\sigma$–finite measure. The set
\[
\left\{ f \in L^1(m) : f \geq 0 \text{ m.e. and } \int f \, dm = 1 \right\}
\]
of all densities in $L^1(m)$ is denoted by $\mathcal{D}$. Recall that a linear operator $P : L^1(m) \to L^1(m)$ is said to be stochastic if $P(\mathcal{D}) \subseteq \mathcal{D}$. By the main result of [LaMy] the semigroups $\mathcal{T}$ which satisfy $\lim_{t \to 0^+} \| T(t)f - (\int f \, dm) f_* \| = 0$, where $f_*$ is strictly positive, form a $\rho$–dense $G_δ$ subset of $\mathfrak{M}_C$ (here $C$ is the set of all stochastic operators on $L^1(m)$). This note may be recognized as a generalization of [LaMy] as we obtain corresponding results for abstract Banach lattices.

A densely defined operator $A_\mathcal{T} x = \lim_{t \to 0^+} \frac{T(t)x - x}{t}$ is called the generator of the semigroup $\mathcal{T}$. Similarly as in [LaMy] we use the Phillips technique of perturbations. Recall (see [G] for all details) that for any $B \in \mathcal{L}(E)$ the operator $A_\mathcal{T} + B$ is a generator of a continuous semigroup. This semigroup is defined by

(i) \[
T_0^B(t) + \sum_{n=1}^{\infty} T_n^B(t) \text{ where } T_0^B(t) = T(t)
\]
and inductively

(ii) \[
T_{n+1}^B(t) = \int_0^t T(t-s)BT_n^B(s)ds.
\]

The above series is convergent in the operator norm and the convergence is uniform on bounded intervals.

If both $\mathcal{T}$ and $B$ are positive then for any positive $\varepsilon > 0$ the semigroup $\mathcal{T}^{\varepsilon,B} = \{ T^{\varepsilon,B}(t) \}_{t \geq 0}$ generated by $A_\mathcal{T} + \varepsilon(B - \text{Id})$ is also positive. It may be easily proved that if $B$ and $\mathcal{T}$ are contractions, then
\[
T^{\varepsilon,B}(t) = e^{-\varepsilon t}(T(t) + \sum_{n=1}^{\infty} \varepsilon^n T_n^B(t))
\]
are contractions too (for this we notice the estimation $\| T_n^B(t) \| \leq \frac{t^n}{n!}$). In order to ensure that we stay in $\mathfrak{M}_C$ we have to impose the following condition:

there exists a compact operator $B \in C$ such that

for any $\varepsilon > 0$, and any semigroup $\mathcal{T} \in \mathfrak{M}_C$ with

the generator $A_\mathcal{T}$, the semigroup generated by $A_\mathcal{T} + \varepsilon(B - \text{Id})$ belongs to $\mathfrak{M}_C$.

(*)
All the sets $\mathcal{M}_C$ considered in this paper are assumed to enjoy (*)

**Remark 1.** The condition (*) is satisfied for many natural classes of operators. For instance it is so if $C$ is the set of all stochastic operators on $L^1(m)$.

(*) is valid also for $E = C(Z)$ where $Z$ is a compact topological Hausdorff space, but now $C$ is the family of all Markov operators (we say that $B$ on $C(Z)$ is markovian if $B \geq 0$ and $B1 = 1$). Using Hahn–Banach theorem we may show that generally, if $C$ consists of those linear contractions $B : E \to E$ for which $Bx_0 = x_0$ for a fixed $x_0$ ($B^*x_0^* = x_0^*$ for some $x_0^* \in E^*$) then (*) holds.

Recall that a semigroup $\{T(t)\}_{t \geq 0}$ (an operator $B$) is quasi–compact if

$$
\lim_{t \to \infty} \text{dist}\{T(t), \mathcal{K}(E)\} = 0 \quad (\lim_{n \to \infty} \text{dist}\{B^n, \mathcal{K}(E)\} = 0),
$$

where $\mathcal{K}(E)$ stands for the ideal of compact operators on $E$ and

$$
\text{dist}\{T(t), \mathcal{K}(E)\} = \inf_{K \in \mathcal{K}(E)} \left\| T(t) - K \right\|.
$$

Asymptotic properties of quasi–compact semigroups are well known. The reader is referred to [N] for a comprehensive review of this subject. For the completeness of the paper we present the following result here, which will be used in the sequel.

**Proposition A.** Let $T = \{T(t)\}_{t \geq 0}$ be a positive semigroup of contractions on a Banach lattice $E$. Then the following conditions are equivalent:

(a) $\lim_{t \to \infty} \sup_{\|x\| \leq 1} \text{dist}\{T(t)x, \mathcal{F}\} = 0$, where $\mathcal{F}$ is a compact subset (contraction) of $E$,

(b) there exist constant $0 \geq C, 0 \leq a < 1$ and a finite-dimensional positive projection $Q$ such that $\left\| T(t) - Q \right\| \leq Ca^t$ for all $t \geq 0$,

(c) for some (for all) $t_0 > 0$ there exists a finite-dimensional projection $Q$ such that $\left\| \frac{1}{N} \sum_{n=0}^{N-1} T(nt_0) - Q \right\| \to 0$,

(d) there exists a compact operator $K$ such that $\left\| T(t_0) - K \right\| < 1$ for some positive $t_0$,

(e) for some (for all) $t_0 \geq 0$ there exists a compact set $\mathcal{F}_0 \subseteq E$ such that $\lim_{n \to \infty} \sup_{\|x\| \leq 1} \text{dist}\{T(nt_0)x, \mathcal{F}_0\} = 0$.

**Proof.** (a) $\implies$ (b) Let $T = T(1)$. By Theorem 2 from [B2] there exists natural $d$ and a finite-dimensional positive projection $Q$ such that
\[ \lim_{n \to \infty} \left\| T^{nd} - Q \right\| = 0. \] Let \( \Omega \) denote the subspace of all limit points of the semigroup \( T(t) \), i.e. \( x \in \Omega \) if and only if \( \lim_{j \to \infty} \| T(t_j)x_0 - x \| = 0 \) for some \( t_j \not\to \infty \) and \( x_0 \in E \). It is not difficult to notice that \( \Omega \) coincides with

\[ \{ T(t)Qx : x \in E, 0 \leq t \leq d \} = \{ QT(t)x : x \in E, 0 \leq t \leq d \} = Q(E). \]

Therefore \( \Omega \) is finite-dimensional. It is proved in [B2] that \( \Omega \) is a Banach lattice with the order inherited from \( E \) and for any \( t \geq 0 \) the operator \( T(t) \) is a lattice isomorphism on \( \Omega \). As a result, for a fixed base \( e_1, \ldots, e_r \) of positive normalized and pairwise orthogonal vectors in \( \Omega \), and for all \( t \) we have \( T(t)e_i = e_{\alpha^t(i)} \), where \( \alpha^t \) is a permutation of \( \{1, 2, \ldots, r\} \). Since the semigroup is continuous we have \( T(t)e_i = e_i \) if \( t \) is sufficiently close to \( 0 \). We conclude that for each \( t > 0 \) the operator \( T(t) \) restricted to \( \Omega \) is the identity, so there is \( Q = T(t)Q \).

Now let \( n \) be such that \( t = nd + t' \) where \( 0 \leq t' < d \). We have

\[ \left\| T(t) - Q \right\| = \left\| T(t')T^{nd} - T(t')Q \right\| \leq \left\| T^{nd} - Q \right\|. \]

Applying Theorem 2.1 ([N] page 343) we may find constants \( C > 0 \) and \( 0 < a < 1 \) such that \( \left\| T(t) - Q \right\| \leq Ca^t \).

(b) \( \implies \) (c) is obvious and (c) \( \implies \) (d) is proved in [Li].

(d) \( \implies \) (e) is a part of Theorem 2 in [B2].

(e) \( \implies \) (a) instantly follows from the estimation

\[ \sup_{\|x\| \leq 1} \text{dist}\{T(t)x, \mathcal{F}\} \leq \sup_{\|x\| \leq 1} \text{dist}\{T(nt_0), \mathcal{F}_0\}, \]

where \( \mathcal{F} = \{ T(s)\mathcal{F}_0 : 0 \leq s \leq t_0 \} \) and \( nt_0 \leq t \leq (n+1)t_0 \).

\[ \square \]

**REMARK 2.** It may be easily checked that each of the above conditions implies \( \frac{1}{L} \int_0^L T(t)dt - Q \) \( \to 0 \) as \( L \to \infty \), where \( Q \) is a finite-dimensional positive projection. The reverse does not hold in general. In fact, let us consider \( E \) to be the Banach lattice \( C(T) \) of all continuous functions on the one-dimensional torus \( T \) and \( T(t)f(x) = f(e^{2\pi it}x) \). For all \( L \geq 1 \) we have

\[ \left\| \frac{1}{L} \int_0^L T(t)dt - \lambda \otimes 1 \right\| \leq \frac{2(L - [L])}{L} \leq \frac{2}{L}. \]
where $|L|$ denotes the largest natural number not exceeding $L$ and $\lambda$ stands for the normalized Haar measure on $T$. This shows that the semigroup $T(t)$ is uniformly ergodic. But $T(1)$ is the identity operator, so the semigroup $\{T(t)\}_{t \geq 0}$ is not quasi-compact.

Immediately from Proposition A, we get

**Corollary 1.** If $\lim_{n \to \infty} \| T(nt_0) - Q \| = 0$ for some $t_0 > 0$ and $Q$ is one-dimensional, then $\lim_{t \to \infty} \| T(t) - Q \| = 0$.

**Remark.** For stochastic semigroups the above result has its version for the strong operator topology. Namely, it is proved in [LaMa] (see Theorem 7.4) that if $\{T(t)\}_{t \geq 0}$ is a stochastic semigroup on $L^1(m)$, such that for some $t_0 > 0$ the iterates $T^n(t_0)$ are convergent in the strong operator topology to $\Lambda \otimes f_*$ (here and in the sequel, $\Lambda$ stands for the integral functional $\Lambda(f) = \int f \, dm$), then $\lim_{t \to \infty} \| T(t)f - \Lambda(f)f_* \| = 0$ for all $f \in L^1(m)$. It is an easy exercise that the strong operator version of Corollary 1 is valid for an arbitrary Banach lattice.

Now recall another auxiliary result, which easily follows from Proposition 2.9 in [N] (page 215). By $\mathcal{M}_C(\mathcal{M}_{C,1})$ we denote the set of all semigroups from $\mathcal{T} \in \mathcal{M}_C$ so that the limit $\lim_{t \to \infty} T(t)$ is finite-dimensional (one-dimensional).

Clearly $\overline{\mathcal{M}_C}$ coincides with the set of all quasi-compact semigroups contained in $\mathcal{M}_C$.

**Proposition B.** Let $C \subseteq \mathcal{L}(E)$ be a fixed, closed in the norm topology set of positive and linear contractions on a Banach lattice $E$. If $C$ satisfies (*) then the set $\overline{\mathcal{M}_C}$ is $\rho$-dense and open in $\mathcal{M}_C$.

**Proof.** Let $B \in C$ be a compact operator such that for all $\varepsilon > 0$ and $T \in \mathcal{M}_C$, the semigroup $\{T^{\varepsilon,B}(t)\}_{t \geq 0}$ belongs to $\mathcal{M}_C$. By [N] this semigroup is quasi-compact. Hence the first part of the proposition is clear. To get the openness of $\overline{\mathcal{M}_C}$, it is sufficient to apply Proposition A (d). In fact, we have the characterization $\overline{\mathcal{M}_C} = \{ T \in \mathcal{M}_C : \text{dist}(T, \mathcal{K}(E)) < 1 \}$.

The next two results are strongly related to [LaMy]. They are included here for the sake of completeness of the paper. We would like to emphasize that the proofs of Corollaries 2 and 3 presented here seem to be slightly shorter and have stronger versions than the corresponding ones in [LaMy].

Let us consider the Banach lattice $L^1(m)$, where $m$ is a $\sigma$-finite measure and $C$ is the set of all stochastic operators on $L^1(m)$. Using (i) and (ii) for
fixed \( f_0 \in \mathcal{D}, \varepsilon > 0 \), and \( \mathcal{T} \in \mathcal{M}_c \) we get
\[
\sup_{f_1, f_2 \in \mathcal{D}} \| T^{\varepsilon, \Lambda \otimes f_0}(t) f_1 - T^{\varepsilon, \Lambda \otimes f_0}(t) f_2 \|
= e^{-\varepsilon t} \sup_{f_1, f_2 \in \mathcal{D}} \| T(t)f_1 - T(t)f_2 \| \leq 2e^{-\varepsilon t} < 2,
\]
where \( t > 0 \) is arbitrary. Combining Proposition A with Theorem 1 from [B3], we obtain \( Q = \lim_{n \to \infty} T^{\varepsilon, \Lambda \otimes f_0}(nt) = \Lambda \otimes f_* \) for some \( f_* \in \mathcal{D} \). Following our Corollary 1 we have \( \lim_{t \to \infty} \left\| T^{\varepsilon, \Lambda \otimes f_0} - \Lambda \otimes f_* \right\| = 0 \). As a result we have obtained:

**Corollary 2.** Let \( \mathcal{C} \) be the set of all stochastic operators on \( L^1(m) \). Then the set \( \mathcal{M}_{\mathcal{C},1} \) is \( \rho \)-dense and open in \( \mathcal{M}_c \).

Since \( m \) is \( \sigma \)-finite we may construct \( m \) almost everywhere strictly positive \( f_0 \in \mathcal{D} \). For any \( f \in \mathcal{D} \), we have
\[
T^{\varepsilon, \Lambda \otimes f_0}(1)f \geq e^{-\varepsilon} \int_0^1 T(t - s)f_0 ds > 0 \text{ a.e.}
\]
Hence invariant density \( f_* = T^{\varepsilon, \Lambda \otimes f_0}(1)f_* \) is strictly positive. Finally we notice that the family \( \mathcal{M}_{\mathcal{C},1,+,\overline{t}} \) of those semigroups \( \mathcal{T} \in \mathcal{M}_c \) which satisfy
\[
\lim_{t \to \infty} \left\| T(t) - \Lambda \otimes f_* \right\| = 0 \text{ with strictly positive } f_* \text{ may be represented as}
\]
\[
\bigcap_{l=1}^{\infty} \bigcap_{j=1}^{\infty} \left( \bigcup_{t > 0} M_{l,j,t} \cap \mathcal{M}_{\mathcal{C},1} \right),
\]
where
\[
M_{l,j,t} = \left\{ \mathcal{T} \in \mathcal{M}_c : \frac{m\{x \in E_j : T(t)f(x) > 0\}}{m(E_j)} > 1 - \frac{1}{l} \text{ for all } f \in \mathcal{D} \right\}
\]
are \( \rho \)-open.

Now we are in a position to formulate the result which corresponds to Theorem 6 obtained by Lasota and Myjak in [LaMy]. We point out that our version is stronger as their convergence is only in the strong operator topology.

**Corollary 3.** Let \( \mathcal{C} \) be the family of stochastic operators on \( L^1(m) \). Then \( \mathcal{M}_{\mathcal{C},1,+} \) is a \( \rho \)-dense \( G_5 \) subset of \( \mathcal{M}_c \).

**Remark 4.** The reader is referred to [I] for Baire category theorems of a single stochastic operator.
Another important class of positive operators if that of Markov operators. Let $Z$ be a compact topological Hausdorff space and $C(Z)$ denote the Banach lattice of all continuous functions on $Z$, with the ordinary supremum norm and the pointwise order. We have already noticed that if $C$ stands for the family of Markov operators on $C(Z)$ then $(*)$ is fulfilled. One-dimensional Markov projections are of the form $\mu \otimes 1$, where $\mu$ belongs to $P(Z)$, the set of all probability (Radon) measures on $Z$. If $\mu \otimes 1$ is a limit of a quasi-compact semigroup $\{T(t)\}_{t \geq 0}$ then $T(t)^* \mu = \mu$ for all $t \geq 0$. Let us recall (see [S]) that a closed set $F \subseteq Z$ is said to be $\mathcal{T}$-invariant if for all $t \geq 0$ and $x \in F$ we have $T(t)^* \delta_x (F) = 1$ (equivalently $T(t)f(x) = 0$ on $F$ if $f \in C(Z)$ satisfies $f \equiv 0$ on $F$). A $\mathcal{T}$-invariant set $F$ is called $\mathcal{T}$-minimal if there is no smaller $\mathcal{T}$-invariant set included in $F$. It is well known (see [S]) that every $\mathcal{T}$-invariant set $F$ always contains at least one $\mathcal{T}$-minimal subset. By [B3] a quasi-compact semigroup $\mathcal{T}$ has only finitely many minimal sets, which coincide with the supports of ergodic $\mathcal{T}$-invariant probability measures ($\mu \in P(Z)$ is said to be ergodic if it is an extreme point of the *weak-*compact and nonempty set

$$P_{\mathcal{T}}(Z) = \{\mu \in P(Z) : T(t)^* \mu = \mu \text{ for all } t \geq 0\}.$$ 

Distinct $\mathcal{T}$-invariant and ergodic probabilities of a quasi-compact semigroup $\mathcal{T}$ have always disjoint supports.

On the other hand, if a quasi-compact semigroup $\mathcal{T}$ of Markov operators on $C(Z)$ has the unique minimal invariant set $M$, then there exists a measure $\mu \in P(Z)$ such that $\lim_{t \to \infty} \|T(t) - \mu \otimes 1\| = 0$. In this case $\text{supp}(\mu) = M$.

It easily follows from invariance that if $M_1$ and $M_2$ are two distinct $\mathcal{T}$-minimal sets then for every pair of probability measures $\nu_i \in P(M_i)$ $i = 1, 2$ we have $\|T(t)^* \nu_1 - T(t)^* \nu_2\| = 2$, for all $t \geq 0$. The last property implies that a quasi-compact semigroup $\mathcal{T}$ is convergent to a one-dimensional projection if and only if there exists $t_0 > 0$ such that

$$\sup_{\nu_1, \nu_2 \in P(Z)} \|T(t_0)^* \nu_1 - T(t_0)^* \nu_2\| < 2,$$

what holds exactly when $P_{\mathcal{T}}(Z)$ is a singleton.

The reader is referred to [B1] and [B2] for detailed informations concerning the structure of minimal invariant sets of quasi-compact Markov operators as well as asymptotic properties of their iterates.

**Corollary 4.** Let $C$ be the set of all Markov operators on $C(Z)$. Then $\overline{M_{C, 1}}$ is $\rho$-dense and open in $M_C$.

**Proof.** The openness follows from the representation

$$\overline{M_{C, 1}} = \bigcup_{t > 0} (M_C \cap \{T : \sup_{\nu_1, \nu_2 \in P(Z)} \|T(t)^* \nu_1 - T(t)^* \nu_2\| < 2\}).$$
which may be easily obtained using Theorem 1 in [B2].

For the denseness we consider the Markov projection \( B_\mu = \mu \otimes 1 \), where \( \mu \) is a probability measure on \( Z \) such that \( \text{supp}(\mu) = M \) is the only \( B_\mu \)-minimal set (for instance we may take \( \mu = \delta_z \) with an arbitrary \( z \in Z \)).

Now, let \( M_1 \) and \( M_2 \) be two \( T^e, B_\mu \)-invariant sets. Note that always, \( T^e, B \)-invariant sets are \( B \)-invariant. Suppose not, and let \( F \) be \( T^e, B \)-invariant but not \( B \)-invariant. We find a nonnegative function \( f \in C(Z) \) satisfying \( f \equiv 0 \) on \( F \), but such that \( Bf(x) = \alpha > 0 \) for some \( x \in F \). By the strong continuity of \( T \) for small \( t_0 > 0 \) we have

\[
T(\tau)BT(s)f(x) > \frac{\alpha}{2} \text{ if } 0 \leq s, \tau < t_0.
\]

This yields

\[
0 = \langle f, [T^e, B(t_0)]^* \delta_x \rangle = T^e, B(t_0)f(x) \\
\geq e^{-\varepsilon t_0} \int_0^{t_0} T(t_0 - s)BT(s)f(x) \, ds > \varepsilon t_0 \frac{\alpha}{2} e^{-\varepsilon t_0}
\]

and we get a contradiction.

Therefore, every two \( T^e, B_\mu \)-invariant sets \( M_1 \) and \( M_2 \) contain the set \( M \), therefore there is exactly one \( T^e, B_\mu \)-minimal set. This implies \( T^e, B_\mu \in M_{C,1} \) and the proof is completed.

\[ \square \]

If \( Z \) is compact and metrizable then similarly as for stochastic semigroups we get:

**Corollary 5.** Let \( C \) be as in Corollary 4. Then the set \( \widehat{M_{C,1,+,1}} \) of those Markov semigroups \( T \) on \( C(Z) \) for which \( \lim_{t \to \infty} T(t) = \mu \otimes 1 \), with \( \text{supp}(\mu) = Z \), is a \( \rho \)-dense \( G_\delta \) subset of \( \widehat{M_{C}} \).

**Proof.** Let \( U_j \) be a countable open base for the topology of \( Z \). We identify \( \widehat{M_{C,1,+,1}} \) as \( \bigcap_{j=1}^{\infty} \bigcup_{\varepsilon > 0} \bigcup_{t > 0} M_{j,\varepsilon,t} \) where

\[
M_{j,\varepsilon,t} = \{ T \in \widehat{M_{C,1}} : \inf_{\nu \in P(Z)} T(t)^* \nu(U_j) > \varepsilon \}.
\]

Clearly \( M_{j,\varepsilon,t} \) are \( \rho \)-open. To show the denseness we may construct semigroups \( T^e, B \), where \( \varepsilon \to 0 \), \( B = \mu \otimes 1 \) and \( \text{supp}(\mu) = Z \).

\[ \square \]
References


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