REMARKS ON POINTS BEING NONWANDERING 
IN A GENERALIZED SENSE (IN DYNAMICAL 
SYSTEMS ON METRIC SPACES)

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The purpose of the present paper is to discuss some properties of points which are nonwandering in a generalized sense, that is which satisfy conditions of the form: \( x \in \hat{J}^+(\pi_+(x)) \) or \( x \in \hat{J}^-(\pi_-(x)) \), where the sets \( \hat{J}^+(M) \) and \( \hat{J}^-(M) \) (the definition introduced in [2] is recalled below) are – in general – essentially larger than the classical prolongational limit sets \( J^+(M) \) and \( J^-(M) \) respectively.

1. First of all, let us recall the fundamental definitions, terminology and notation used below (see for instance [1]).

Let \( (X, R; \pi) \) be a dynamical system, with \( (X, \rho) \) being a metric space. This means that \( \pi: R \times X \longrightarrow X \) is a continuous mapping such that

\[
\pi(0, x) = x \quad \text{for } x \in X \quad \text{and} \quad \pi(s, \pi(t, x)) = \pi(s + t, x) \quad \text{for } x \in X \quad s, t, \in R.
\]

For a given point \( x \in X \) we put

\[
(1.0) \quad \pi_+(x) := \{\pi(t, x): t \geq 0\}, \quad \pi_-(x) := \{\pi(t, x): t \leq 0\},
\]

and

\[
(1.1) \quad \pi(x) := \pi_+(x) \cup \pi_-(x),
\]

\[
(1.2) \quad \Lambda^+(x) := \{y: \text{there is a sequence } \{t_n\} \text{ of real numbers such that } t_n \longrightarrow \infty \text{ and } \pi(t_n, x) \longrightarrow y \text{ as } n \rightarrow \infty\},
\]

\[
(1.3) \quad J^+(x) := \{y: \text{there is a sequence } \{t_n\} \text{ of real numbers and } \{x_n\} \text{ of elements of } X \text{ such that } t_n \longrightarrow \infty, x_n \longrightarrow x \text{ and } \pi(t_n, x_n) \longrightarrow y \text{ as } n \rightarrow \infty\},
\]
The sets $\pi_+(x)$ ($\pi_-(x)$), $\pi(x)$, $\Lambda^+(x)$, $J^+(x)$ are called: the positive (negative) semitrajectory of $x$, the trajectory of $x$, the positive limit set of $x$ and the positive prolongation limit set of $x$, respectively. The negative limit set $\Lambda^-(x)$ and the negative prolongational limit set $J^-(x)$ of $x$ are defined by formulae analogous to (1.2) and (1.3) respectively, where the condition "$t_n \to \infty$" is replaced by "$t_n \to -\infty$".

A point $x \in X$ is said to be nonwandering if $x \in J^+(x)$ (this is equivalent to $x \in J^-(x)$).

The mapping $\pi^x: \mathbb{R} \ni t \mapsto \pi^x(t) := \pi(t, x) \in X$ is called the motion of the point $x$. The motion $\pi^x$ is said to be positively (negatively) Poisson stable if and only if $x \in \Lambda^+(x)$ ($x \in \Lambda^-(x)$). The motion $\pi^x$ is said to be strongly uniformly positively (negatively) Lyapunov stable if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(s \in \mathbb{R}, \rho(y, \pi(s, x)) < \delta) \implies \left( \rho(\pi(t, y), \pi(t + s, x)) < \varepsilon \right)$$

for $t \geq 0$ ($t \leq 0$).

We have the following almost trivial

**Proposition 1.** If $\pi^x$ is strongly uniformly positively (negatively) Lyapunov stable, \{\{t_n\}, \{s_n\}\} are sequences of real nonnegative (nonpositive) numbers, \{x_n\} is a sequence of elements of $X$ such that

$$\rho(x_n, \pi(s_n, x)) \to 0 \quad \text{as} \quad n \to \infty$$

then

$$\rho(\pi(t_n, x_n), \pi(t_n + s_n, x)) \to 0 \quad \text{as} \quad n \to \infty.$$

Let $M$ be a nonempty subset of $X$. We put (see [2]):

$$\hat{\mathcal{J}}^+(M) := \left\{ y : \text{there are sequences } \{t_n\} \text{ of real numbers} \right\}$$

(1.5)

$$\quad \text{and } \{x_n\} \text{ of elements of } X \text{ such that } t_n \to \infty,$$

$$\rho(x_n, M) \to 0 \text{ and } \pi(t_n, x_n) \to y \text{ as } m \to \infty \right\}.$$

Here $\rho(x_n, M) = \inf\{\rho(x_n, z) : z \in M\}$. We define also $\hat{\mathcal{J}}^-(M)$ by a similar formula, namely by substituting in (1.5) the condition "$t_m \to -\infty$" in the place of "$t_m \to \infty$". Fundamental (and mostly: quite elementary) properties of the sets $\hat{\mathcal{J}}^+(M)$ and $\hat{\mathcal{J}}^-(M)$ are discussed in [2]; there are also observations concerning some essential differences between $\hat{\mathcal{J}}^+(M)$ and $J^+(M)$ as well as between $\hat{\mathcal{J}}^-(M)$ and $J^-(M)$.

It seems to be natural, that we will say that a point $x \in X$ is nonwandering in the generalized sense if $x \in \hat{\mathcal{J}}^+(\pi_+(x)) \cap \hat{\mathcal{J}}^-(\pi_-(x))$. 
2. THEOREM 1. Assume that \( x \in X \) is such that

(i) the motion \( \pi^x \) is strongly uniformly positively Lyapunov stable,

(ii) \( x \in \hat{J}^+(\pi_+(x)) \).

Then \( \pi^x \) is positively Poisson stable.

**PROOF.** We have

\[
(2.1) \quad x = \lim \pi(t_n, x_n)
\]

for some sequences \( \{t_n\} \) of real numbers and \( \{x_n\} \) of elements of the space \( X \) such that

\[
(2.2) \quad t_n \rightarrow \infty
\]

and

\[
(2.3) \quad \rho(x_n, \pi_+(x)) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

The last condition means that there is a sequence of nonnegative numbers \( \{s_n\} \) such that

\[
(2.4) \quad \rho(x_n, \pi(s_n, x)) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

This implies (see Proposition 1) that

\[
(2.5) \quad \rho(\pi(t_n, x_n), \pi(t_n + s_n, x)) \rightarrow 0.
\]

The conditions (2.1) and (2.5) give

\[
x = \lim \pi(t_n + s_n, x)
\]

and because of the obvious relation: \( t_n + s_n \rightarrow \infty \), we get finally

\[
x \in \Lambda^+(x)
\]

which finishes the proof.

**REMARK 1.** It is clear that if \( \pi^x \) is strongly uniformly negatively stable and \( x \in \hat{J}^-(\pi_-(x)) \) then \( x \) is negatively Poisson stable; the proof is obviously similar to that of Theorem 1.

**REMARK 2.** If \( X = \mathbb{R}^2 \), then the Poisson stability implies the periodicity of \( \pi^x \); so in that case the assumptions (i) and (ii) give the periodicity of \( \pi^x \).
3. **Theorem 2.** If \( J^+(x) \cap \Lambda^-(x) \neq \emptyset \ (J^-(x) \cap \Lambda^+(x) \neq \emptyset) \) then \( x \in \hat{J}^-(\pi_-(x)) \ (x \in \hat{J}^+(\pi_+(x))). \)

**Proof.** Suppose that \( y \in J^+(x) \cap \Lambda^-(x) \). Hence there are sequences \( \{t_n\}, \{s_n\} \) of real numbers and \( \{x_n\} \) of elements of \( X \) such that \( t_n \to \infty, s_n \to -\infty, \)

\[
y = \lim \pi(t_n, x_n) = \lim \pi(s_n, x)
\]

and

\[
x = \lim x_n.
\]

Putting

\[
y_n := \pi(t_n, x_n)
\]

we get clearly

\[
x_n = \pi(-t_n, x_n)
\]

and, because of (3.1)

\[
\rho(y_n, \pi(s_n, x)) \to 0,
\]

which gives in particular

\[
\rho(y_n, \pi_-(x)) \to 0 \text{ as } n \to \infty.
\]

So

\[
x = \lim \pi(-t_n, y_n)
\]

where \( \{y_n\} \) is a sequence satisfying (3.6), and \( -t_n \to -\infty \); this means that

\[
x \in J^-(\pi_-(x)).
\]

A similar ("symmetric") reasoning proves that \( x \in J^+(\pi_+(x)) \) if \( J^-(x) \cap \Lambda^+(x) \neq \emptyset \). The proof is completed.

**Corollary 1.** If \( \Lambda^+(x) \cap \Lambda^-(x) \neq \emptyset \) then \( x \) is nonwandering in the generalized sense.

**Corollary 2.** If \( \pi^x \) is strongly uniformly positively Lyapunov stable and \( \Lambda^+(x) \cap \Lambda^-(x) \neq \emptyset \) then \( \pi^x \) is positively Poisson stable.
Corollary 3. If $\pi^x$ is strongly positively uniformly Lyapunov stable and $x$ does not belong to $\hat{J}^+(x)$, then $\Lambda^+(x) \cap \Lambda^-(x) \neq \emptyset$.

Theorem 3. If $x \in \hat{J}^+(\pi_+(x))$ ($x \in \hat{J}^-(\pi_-(x))$) and $\pi_+(x)$ ($\pi_-(x)$) is compact, then $J^-(x) \cap \pi_+(x) \neq \emptyset$ ($J^+(x) \cap \pi_-(x) \neq \emptyset$).

Proof. Assume that $x \in \hat{J}^+(\pi_+(x))$ and that the closure of the positive semitrajectory is compact. So

$$x = \lim \pi(t_n, x_n) \tag{3.7}$$

where $t_n \to \infty$ and

$$\rho(y_n, \pi_+(x)) \to 0. \tag{3.8}$$

Let us put

$$w_n := \pi(t_n, y_n). \tag{3.9}$$

We have obviously

$$y_n = \pi(-t_n, w_n) \tag{3.10}$$

and

$$\rho(y_n, \pi(s_n, x)) \to 0 \tag{3.11}$$

for some sequence $\{s_n\}$ of nonnegative numbers. Since $\pi_+(x)$ is compact, we may assume that the sequence $\{\pi(s_n, x)\}$ is convergent to some element $y$ of $\pi_+(x)$. Thus, because of (3.11), the sequence $\{y_n\}$ is also convergent to $y$. Since $w_n \to x$ and $-t_n \to -\infty$, we get by virtue of (3.10), the relation $y \in J^-(x)$, and then finally

$$J^-(x) \cap \pi_+(x) \neq \emptyset.$$

A similar reasoning gives the alternative implication: $x \in \hat{J}^-(\pi_-(x))$ and compactness of $\pi_-(x)$ imply

$$J^+(x) \cap \pi_-(x) \neq \emptyset.$$

The proof is completed.
COROLLARY 4. If $\pi(x)$ is compact and $x \in \hat{J}^+(\pi_+(x)) \cap \hat{J}^-(\pi_-(x))$ then $J^+(x) \cap \pi_-(x) \neq \emptyset$ and $J^-(x) \cap \pi_+(x) \neq \emptyset$.

COROLLARY 5. If $\pi_+(x)$ $(\pi_-(x))$ is compact, $x \in \hat{J}^+(\pi_+(x))$ $(x \in \hat{J}^-(\pi_-(x)))$ and $x$ is a wandering point (which means that $x \notin J^+(x)$ and $x \notin J^-(x)$) then $J^-(x) \cap \Lambda^+(x) \neq \emptyset$ $(J^+(x) \cap \Lambda^-(x) \neq \emptyset)$.

COROLLARY 6. If $\pi(x)$ is compact, $x$ is wandering point in the generalized sense but $x$ is not wandering, then

$$J^-(x) \cap \Lambda^+(x) \neq \emptyset \quad \text{and} \quad J^+(x) \cap \Lambda^-(x) \neq \emptyset.$$  

COROLLARY 7. Assume that $x$ is a wandering point and that $\pi_+(x)$ $(\pi_-(x))$ is compact. Then the conditions:

(ij) $J^-(x) \cap \Lambda^+(x) \neq \emptyset$ \quad $(J^+(x) \cap \Lambda^-(x) \neq \emptyset)$

and

(ii) $x \in \hat{J}^+(\pi_+(x))$ \quad $(x \in \hat{J}^-(\pi_-(x)))$

are equivalent.

COROLLARY 8. If $\pi(x)$ is compact and $x$ is a wandering point, then the conditions

(k) $J^-(x) \cap \Lambda^+(x) \neq \emptyset$ \quad and \quad $J^+(x) \cap \Lambda^-(x) \neq \emptyset$.

and

(kk) $x$ is nonwandering in the generalized sense,

are equivalent.

4. Some examples. Consider four dynamical systems on $X = \mathbb{R}^2$ having trajectories presented below in Pictures I – IV. In each case the points $x$ and $y$ (as well as the point $z$ in Pictures III and IV) are supposed to be elements of the unit circle, denoted by $C$.

In these four examples we have the following properties

I. $\Lambda^+(x) = \Lambda^-(x) = \{y\}, \ x \in J^+(x) = J^-(x) = \pi(x) \cup \{y\} = \hat{J}^+(\pi_+(x))$ \quad $= \hat{J}^-(\pi_-(x)) = \pi(x) = C$.

II. $\Lambda^+(x) = \Lambda^-(x) = J^+(x) = J^-(x) = \{y\}$, and of course $x \notin J^+(x)$, \quad $x \notin J^-(x)$.

III. $\Lambda^+(x) \cap \Lambda^-(x) = \emptyset$, $x \notin J^+(x) = J^-(x) = \Lambda^+(x) = \{y\}$, $x \in \hat{J}^+(\pi_+(x)) = \emptyset$ but $x \notin J^-(\pi_-(x))$.

IV. $x \in \hat{J}^+(\pi_+(x)) \cap \hat{J}^-(\pi_-(x)) \setminus (J^+(x) \cup J^-(x))$, $\Lambda^+(x) = \{y\}$, $\Lambda^-(x) = \{z\}$ and so $\Lambda^+(x) \cap \Lambda^-(x) = \emptyset$, $J^-(x) \cap \Lambda^+(x) = \{y\}$, $J^+(x) \cap \Lambda^-(x) = \{z\}$.  

References


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