CHEBYSHEV POLYNOMIALS
ON EQUIPOTENTIAL CURVES
OF A QUADRATIC JULIA SET

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Abstract. We prove that the Chebyshev polynomial of degree $2^n$ ($n \geq 0$) on the Julia set $J_{\lambda}$ for a quadratic polynomial $T_{\lambda}(z) = (z - \lambda)^2$, where $0 \leq \lambda \leq 2$, is also the Chebyshev polynomial on any equipotential curve $\Gamma_R$ ($R > 1$) in the unbounded component of $\mathbb{C} \setminus J_{\lambda}$. This class of Julia sets contains the unit circle and the real line segment $[0, 4]$, for which the mentioned property of the Chebyshev polynomials has been already known to hold (see e.g. [Fab]).

0. Introduction. For a compact subset $K$ of the complex plane $\mathbb{C}$ with at least $n$ elements there exists a unique complex polynomial $P_n$ of degree $n$ with leading coefficient 1 such that

$$\max_{z \in K} |p_n(z)| = \inf \left\{ \max_{z \in K} |z^n + a_1 z^{n-1} + \ldots + a_n| : a_1, \ldots, a_n \in \mathbb{C} \right\}.$$  

The polynomial $P_n$ is called the n-th Chebyshev polynomial for the set $K$. For example, if $K$ is the unit circle $\{z \in \mathbb{C}: |z| = 1\}$, then $P_n(z) = z^n$. If $K$ is the segment $[0, 4]$ on the real axis, then $P_n$ is the classical n-th Chebyshev polynomial, that is $P_n(z - 2) = w^n + \frac{1}{w^n}$, where $z = w + \frac{1}{w} + 2$, $|w| > 1$. In both these cases the polynomials with minimal maximum norm on $K$ are also Chebyshev polynomials for equipotential curves in $\hat{\mathbb{C}} \setminus K$, i. e. for the level lines of the function $\Phi$ that maps conformally the (unbounded component of) complement of $K$ onto the exterior of the unit circle. These lines have the form

$$\Gamma_R = \{z: |\Phi(z)| = R\}, \quad R > 1.$$
For the unit circle this observation is straightforward, because \( P_n(z) = z^n \) is the \( n \)-th Chebyshev polynomial for any circle \( \{ z \in \mathbb{C} : |z| = R \} \), which is the level curve in question. For the real segment \([0, 4]\) the equipotential curves are ellipses with foci 0 and 4. It was proved by Faber [Fab] that classical Chebyshev polynomials are also polynomials with minimal uniform norm on such curves.

One may ask a question if there are any other compact planar sets \( K \) such that the Chebyshev polynomials on \( K \) (or at least a subsequence of them) are also the Chebyshev polynomials on equipotential curves in \( \hat{\mathbb{C}} \setminus K \). Fischer [Fis] has shown that this is the case when \( K \) is the union of certain two disjoint real segments. Here we study the problem for Julia sets of some quadratic polynomials (see the definition below). We show that they provide new examples of sets with the desired property, among which the unit circle and the real line segment \([0, 4]\) appear as special cases.

1. Preliminaries. Let a complex quadratic polynomial \( T_\lambda : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) of the form \( T_\lambda(z) = (z - \lambda)^2 \) be given. (In fact any complex quadratic polynomial can be conjugated by suitable transformations to a polynomial \( T_\lambda(z) = (z - \lambda)^2 \).)

The symbol \( T_\lambda^{\circ n} \) denotes the \( n \)-th iterate of the polynomial \( T_\lambda \), namely

\[
T^{\circ 0}(z) = z \quad \text{and} \quad T^{\circ n}(z) = T(T^{\circ (n-1)}(z)) \quad \text{for} \quad n \geq 1 \quad (z \in \hat{\mathbb{C}}).
\]

As \( T_\lambda(\infty) = \infty \) and \( T_\lambda'(\infty) = 0 \), the set

\[
A(\infty) = \{ z \in \hat{\mathbb{C}} : T_\lambda^{\circ n}(z) \rightarrow \infty \}
\]

is a non-empty open subset of \( \hat{\mathbb{C}} \). It is also connected and completely invariant under \( T_\lambda \), i.e. \( T_\lambda^{-1}(A(\infty)) = T_\lambda(A(\infty)) = A(\infty) \). Its boundary \( J_\lambda \) is the Julia set for the polynomial \( T_\lambda \).

The set \( J_\lambda \) is a non-empty compact perfect subset of the complex plane \( \mathbb{C} \). The term \( \lambda \) in the polynomial \( T_\lambda \) dictates further topological and geometrical properties of \( J_\lambda \). For example, it is known ([Bea]) that for \(-\frac{1}{4} \leq \lambda \leq 2 \) (\( \lambda \) real) \( J_\lambda \) is connected. In this paper we restrict our attention to \( 0 \leq \lambda \leq 2 \).

It can be proven that for any \( T_\lambda \) there exists a mapping \( \Phi_\lambda \) analytic in a neighborhood of infinity such that

\[
(1.1) \quad \Phi_\lambda(T_\lambda(z)) = [\Phi_\lambda(z)]^2
\]

(cf. [Bea, th. 6.10.1] or [CaG th. II.4.1]).
The connectedness of the Julia set $J$ for any polynomial $T$ of degree $d > 1$ is equivalent to the fact that there are no finite critical points of $T$ in $A(\infty)$ (i.e. $T'(z) \neq 0$ at every $z \in A(\infty)$). Thus for $0 \leq \lambda \leq 2$ the mapping $\Phi_\lambda$ can be extended in a unique way to an analytic isomorphism (which we shall still denote by $\Phi_\lambda$) from $A(\infty)$ onto $B_0 := \{w: |w| > 1\}$. Consequently, there exist equipotential curves defined as

$$\Gamma_R = \{z: |\Phi_\lambda(z)| = R\}, \quad R > 1$$

which are closed analytic curves surrounding $J_\lambda$.

The polynomial $T_\lambda$ maps twofold the curve $\{|\Phi_\lambda(z)| = R\}$ onto $\{|\Phi_\lambda(z)| = R^2\}$ (see [CaG]).

One can also define the function $F_\lambda$ inverse to $\Phi_\lambda$,

$$F_\lambda: B_0 \rightarrow A(\infty),$$

for which the following relation holds:

$$F_\lambda(w^2) = T_\lambda(F_\lambda(w)).$$

Let us consider the sequence of polynomials $\{P_{2^k}\}_{k=0}^\infty$ which satisfy the following equations (see [BGH 2]):

$$P_1(z) = z - \lambda, \quad k = 0, 1, 2, \ldots$$

$$P_{2^k}(z) = P_{2^{k-1}}(T_\lambda(z)), \quad k = 1, 2, \ldots$$

Thus every $P_{2^k}$ is of degree $2^k$ and has leading coefficient equal to 1.

Barnsley et al. (in [BGH 2]) have proven that for all $\lambda \in \mathbb{C}$ the polynomial $P_{2^k}$ is the $2^k$-th Chebyshev polynomial on $J_\lambda$ ($k = 0, 1, 2, \ldots$). Its maximum norm on $J_\lambda$ equals $\tau_\lambda = \frac{1}{2} + \sqrt{\frac{1}{4} + |\lambda|}$.

**Example.** If $\lambda = 0$, then $J_\lambda = \{z = e^{i\Theta}\}$, $\Phi_\lambda(z) = z$, $F_\lambda(w) = w$ and $\Gamma_R = \{z \in \mathbb{C}: |z| = R\}, R > 1$ (an equipotential curve is a circle with center at the origin).

If $\lambda = 2$, then $J_\lambda = [0, 4]$, $F_\lambda(w) = w + \frac{1}{w} + 2$, $\Phi_\lambda(z) = \frac{z - 2 + \sqrt{(z-2)^2 - 4}}{2}$ (where the branch of the square root is taken so that $\Phi_\lambda$ maps the complement of the segment $[0, 4]$ onto the exterior of the unit disc), and

$$\Gamma_R = \left\{z = x + iy \in \mathbb{C}: x - 2 = \left(R + \frac{1}{R}\right) \cos \varphi, \quad y = \left(R - \frac{1}{R}\right) \sin \varphi, \quad \varphi \in [0, 2\pi], \quad R > 1\right\}$$
(an equipotential curve is an ellipse with foci 0 and 4).

No explicit formulas for functions $\Phi_\lambda$ and $F_\lambda$ corresponding to other $\lambda \in (0, 2)$ seem to be known. Nevertheless, the functional equations (1.1) and (1.2) satisfied respectively by $\Phi_\lambda$ and $F_\lambda$ enable us to compute exactly the norms of the $P_{2^k}$, $k = 0, 1, 2, \ldots$, on any equipotential curve $\Gamma_R = F_\lambda\left(\{z: |z| = R\}\right)$, $R > 1$.

2. The norm of the polynomial $P_{2^k}$ on an equipotential curve $\Gamma_R$. In order to compute the norm of the polynomial $P_{2^k}$ ($k = 0, 1, 2, \ldots$) on an equipotential curve $\Gamma_R$ we shall approximate the function $F_\lambda$ by analytic functions which converge uniformly to $F_\lambda$ on compact subsets of $B_o$. We shall take advantage of two approximating sequences of analytic functions constructed by Barnsley et al. in [BGH 1].

**Lemma 2.1** ([BGH 1], Lemmas 3 and 5, Theorems 6 and 8).

Let $\lambda \in [0, 2]$, and $r_\lambda = \sqrt{\lambda + \frac{1}{4} + \frac{1}{2}}$.

a) Let $f_0(z) = \lambda + r_\lambda^2 \cdot z$.

We define iteratively analytic functions $f_n(z) = \lambda + \sqrt{f_{n-1}(z^2)}$.

b) Let $h_0(z) = \lambda + \frac{r_\lambda}{2} \left(z + \frac{1}{z}\right)$.

We define iteratively analytic functions $h_n(z) = \lambda + \sqrt{h_{n-1}(z^2)}$.

The sequences $\{f_n\}$ and $\{h_n\}$ converge uniformly to the function $F_\lambda$ on compact subsets of $B_o$.

The approximation of $F_\lambda$ as above allows us to calculate the exact value of the norm of $P_{2^k}$ on an equipotential curve.

**Theorem 2.1.** Let $0 \leq \lambda \leq 2$ and $R > 1$. The maximum norm of the polynomial $P_{2^k}$ on the level curve $\Gamma_R = \{z \in \mathbb{C}: |\Phi_\lambda(z)| = R\}$ equals

\[
\max_{z \in \Gamma_R} |P_{2^k}| = F_\lambda\left(R^{2^k}\right) - \lambda.
\] (2.1)

**Proof.**

a) Let $k = 0$. By the definitions of $\Gamma_R$ and $F_\lambda$, $\max |z - \lambda| = \max |F_\lambda(w) - \lambda|$. $|w| = R$.

Note that $F_\lambda(R) - \lambda = \lim_{n \to \infty} f_n(R) - \lambda$ is a nonnegative real number. Moreover, the critical point $\lambda$ of the polynomial $T_\lambda$ does not belong to $A(\infty)$, so it cannot be the value of $F_\lambda(R)$ if $R > 1$. It follows that $F_\lambda(R) - \lambda$ is strictly positive and $\max |z - \lambda| \geq F_\lambda(R) - \lambda$. 
To prove the reverse inequality, observe that for $0 \leq \lambda \leq 2$, $r_\lambda \geq 1$. So by the definition of the function $f_\omega$ we have the following estimates on $\{w: |w| = R\}$:

\begin{equation}
|f_\omega(w^2)| \leq \lambda + r^2_\lambda R^2 \leq \left(2 + \frac{1}{R^2}\right) + r^2_\lambda R^2 \leq r^2_\lambda \left(R + \frac{1}{R}\right)^2
\end{equation}

and

\begin{equation}
|f_1(w) - \lambda| = \sqrt{|f_\omega(w^2)|} \leq r_\lambda \cdot \left(R + \frac{1}{R}\right) = 2\left(h_\omega(R) - \lambda\right).
\end{equation}

Next,

\[|f_1(w^2)| \leq |f_1(w^2) - \lambda| + \lambda \leq 2h_\omega(R^2) - \lambda \leq 2h_\omega(R^2),\]

so

\begin{equation}
|f_2(w) - \lambda| = \sqrt{|f_1(w^2)|} \leq \sqrt{2} \sqrt{h_\omega(R^2)} = \sqrt{2}\left(h_1(R) - \lambda\right).
\end{equation}

By induction in $n$, we prove that

\begin{equation}
|f_n(w^2)| \leq |f_n(w^2) - \lambda| + \lambda \leq |f_n(w^2) - \lambda| + (2)^{2-(n-1)} \cdot \lambda
\end{equation}

\[\leq (2)^{2-(n-1)} \cdot h_{n-1}(R^2),\]

which by the definition of $f_n$ and $h_n$ implies that

\begin{equation}
|f_{n+1}(w) - \lambda| = \sqrt{|f_n(w^2)|} \leq (2)^{2-n} \cdot \sqrt{h_{n-1}(R^2)}
\end{equation}

\[= (2)^{2-n} \cdot (h_n(R) - \lambda).\]

As $n$ tends to infinity, the left-hand side of (2.5) tends to $|F_\lambda(w) - \lambda|$, while the right-hand side of (2.5) tends to $F_\lambda(R) - \lambda$, which proves the assertion for $k = 0$.

b) Let $k > 0$. As $T_\lambda$ maps $\Gamma_R$ twofold onto $\Gamma_{R^2}$, by equation (1.4) we have

\begin{equation}
\max_{z \in \Gamma_R} |P_{2k}(z)| = \max_{z \in \Gamma_R} |P_{2k-1}(T(z))| = \max_{z \in \Gamma_{R^2}} |P_{2k-1}(w)|.
\end{equation}

Induction in $k$ gives

\begin{equation}
\max_{z \in \Gamma_R} |P_{2k}(z)| = F_\lambda((R^2)^{2k-1}) - \lambda = F_\lambda(R^{2k}) - \lambda.
\end{equation}

\[\square\]

Let us denote by $\Sigma(T_\lambda)$ the group of symmetries of the Julia set $J_\lambda$ for a polynomial $T_\lambda(z) = (z - \lambda)^2$. 
Lemma 2.2 [Bea, th. 9.5.4].

a) $\Sigma(T_{\lambda})$ is a group of rotations about the point $\lambda$. $\Sigma(T_{\lambda})$ is infinite if and only if $\lambda = 0$ (it is then the group of all rotations about 0). When $\lambda \neq 0$, $\Sigma(T_{\lambda}) = \{I, \sigma\}$, where $\sigma(z) = 2\lambda - z$.

b) $\sigma \in \Sigma(T_{\lambda})$ if and only if it commutes with the mapping $\Phi_{\lambda}$.

It follows that the equipotential curves in the unbounded component of the complement of $J_{\lambda}$ are symmetric about $\lambda$. For this reason the polynomial $P_1(z) = z - \lambda$ attains its maximum modulus on the curve $\Gamma_R$ at $F_{\lambda}(R)$ as well as at $2\lambda - F_\lambda(R)$. (There may be more extremal points for $P_1$ on $\Gamma_R$, but we do not take them here into consideration.) Furthermore, the polynomial $P_{2^k}$ ($k \geq 0$) attains its maximum modulus on $\Gamma_R$, equal to $c := F(R^{2^k}) - \lambda$, at each root of $T^{ok}(z) = \lambda + c$ and $T^{ok}(z) = \lambda - c$. These roots lie on $\Gamma_R$, since $T^{on}(z)$ maps $\Gamma_R$ $2^k$-fold onto $\Gamma_{R^{2^n}}$. □

Bearing this information in mind, we can now repeat the argument used by Barnsley et al. in [BGH 2] in order to obtain the following result:

Theorem 2.2. Let $0 \leq \lambda \leq 2$ and $R > 1$. The polynomial $P_{2^n}$ ($n \geq 0$) is the $(2^n)$th-degree Chebyshev polynomial on the equipotential curve $\Gamma_R$ in the unbounded component of the complement of the Julia set $J_{\lambda}$.

Proof. Let $q$ be a polynomial of degree $2^n$ with the leading coefficient one. We shall show that if $z$ is one of the roots of $T^{on}(z) = \lambda \pm c$, then $|q(z)| \geq c$.

We can represent $q$ as the sum

$$(2.9) \quad q(z) = P_{2^n}(z) + \sum_{j=0}^{2^n-1} a_j U_j(z),$$

where $a_j$ are complex coefficients. The polynomial $U_j$ of degree $j$ equals $P_{2^k}$ if $j = 2^k$, $k = 0, 1, \ldots, n - 1$ and $U_0 \equiv 1$. If $j \neq 2^k$, $k = 0, 1, \ldots, n - 1$, then we take $U_j$ to be the product $P_{2^k_1}(z) \ldots P_{2^k_r}(z)$, where $r \geq 2$ and $j = 2^{k_1} + \ldots + 2^{k_r}$. In this manner we obtain polynomials of all degrees $j = 0, 1, \ldots, 2^n - 1$, so the $U_j$’s will form a basis in the space of polynomials of degree at most $2^n - 1$.

Let us now define the index of a polynomial $H$ of degree $\leq 2^n - 1$. It is the largest integer $k \leq n$ such that $H$ can be expressed as a polynomial in $T^{ok}(z)$. (The index of $H$ is $n$ if and only if $H$ is constant.) We can regroup the terms in (2.9) as follows:

$$(2.10) \quad q(z) = P_{2^n}(z) + \sum_{k=0}^{n} q_k(z),$$
where \( q_k(z) \) is the sum of all terms with index \( k \). Note that for \( k < n \), 
\( q_k(z) = P_{2^k}(z) \cdot (\text{terms with index } > k) \). Finally, let 

\[
(2.11) \quad L_s(z) = P_{2^n}(z) + \sum_{k=n-s}^{n} q_k(z), \quad s = 0, 1, \ldots, n.
\]

Then \( L_n(z) = q(z) \).

Actually, we prove the following: for each \( k = 0, 1, \ldots, n \) there are numbers 
\( s_j \in \{ -1, +1 \}, j = 0, 1, \ldots, k \) such that for any \( s_j \in \{ -1, +1 \}, j = k+1, k+2, \ldots, n \) we have 
\[ |L_k(z)| \geq c, \text{ where } z = \lambda + s_n \sqrt[1]{\lambda + s_{n-1} \sqrt[1]{\lambda + \ldots + s_1 \sqrt[1]{\lambda + s_0 c}}}. \]

In particular, for \( k = n \) there is a root \( z \) of \( T^{2^n}(z) = \lambda \pm c \) such that \( |q(z)| \geq c \).

Let \( k = 0 \). With all possible choices of \( s_o, s_1, \ldots, s_n \in \{ -1, +1 \} \) we have

\[
(2.12) \quad |L_0(z)| = |P_n(z) + a_o| = |s_0 c + a_o| = |c + s_o a_o|.
\]

Since \( |c - a_o| \) and \( |c + a_o| \) cannot both be less than \( c \), we can choose 
\( s_0 \in \{ -1, +1 \} \) so that \( |c + s_0 a_o| \geq c \).

Suppose now that the claim is true for \( k = m - 1 < n \).

Let \( s_o, s_1, \ldots, s_{m-1} \) be as in the statement for \( k = m - 1 \).

For \( z \) as above, we have

\[
(2.13) \quad q_{n-m}(z) = P_{2^{n-m}}(z) \cdot (\text{terms with index } > n - m)
= P_{2^{n-m}}(z) \cdot H(T^{n-m+1}(z)),
\]

where \( H \) is a polynomial.

Thus

\[
(2.14) \quad q_{n-m}(z) = (s_m \sqrt[1]{\lambda + s_{m-1} \sqrt[1]{\lambda + \ldots + s_1 \sqrt[1]{\lambda + s_0 c}}})
\times H(\lambda + s_{m-1} \sqrt[1]{\lambda + s_{m-2} \sqrt[1]{\lambda + \ldots + s_1 \sqrt[1]{\lambda + s_0 c}}}).
\]

The \( s_o, s_1, \ldots, s_{m-1} \) being already fixed, \( q_{n-m}(z) \) is determined up to a sign.
We can choose \( s_m \in \{ -1, +1 \} \) so that

\[
(2.15) \quad |L_m(z)| = |L_m(z) + q_{n-m}(z)| \geq |L_{m-1}(z)|,
\]

and \( |L_{m-1}(z)| \geq c \) by the inductive hypothesis.
Thus for any polynomial $q$ of degree $2^n$ with leading coefficient one, we have

\begin{equation}
\max_{w \in \Gamma_R} |q(w)| \geq \max_{w \in \Gamma_R} |P_{2^n}(w)|,
\end{equation}

which proves our result.

\[ \square \]

**Remark.** The case of the Chebyshev polynomials on equipotential curves in the complement of two disjoint real line segments (studied by Fischer) has nothing in common with our examples, because the Julia set $J_{\lambda}$ for a quadratic polynomial $T_{\lambda}(z) = (z - \lambda)^2$ is either connected or totally disconnected.

**References**


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