COMMUTATORS OF HOMEOMORPHISMS
OF A MANIFOLD

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Abstract. It is shown that the identity component of the group of all homeomorphisms of a manifold with boundary is perfect, i.e. equal to its commutator subgroup. Differences between homeomorphism and diffeomorphism groups are exhibited.

1. Introduction. For an arbitrary group $G$ recall the following definitions. $G$ is called simple if it has no nontrivial normal subgroups. For any $g, h \in G$ their commutator is defined as $[g, h] = ghg^{-1}h^{-1}$, and all the commutators generate the commutator subgroup $[G, G]$. The quotient group $G/[G, G]$ is said to be the abelianization of $G$. Next $G$ is called perfect if its abelianization is trivial. It is immediate that any nonabelian simple group is perfect.

When considering homeomorphism or diffeomorphism groups, the problem of the simplicity and perfectness of their identity component is of interest (see [1], [5], [6], [7], [10], [11], [12], [14], [16] and references therein). In view of a theorem of Epstein [5] the problem of simplicity in a large class of transitive homeomorphism groups amounts to showing their perfectness. Notice that a nontransitive group of homeomorphisms cannot be simple (with the exception of some non-generic cases) as the subgroup of all homeomorphisms fixing points of some orbit is normal and (usually) nontrivial. Nevertheless the question of perfectness of a nontransitive homeomorphism group is still valid and difficult to answer (see e.g. the author's paper [14]). We shall deal with homeomorphism groups on a manifold with boundary, the groups which are obviously nontransitive.

Let $M$ be an $n$-dimensional second countable topological manifold with or without boundary. By $\mathcal{H}(M)$ we denote the group of all homeomorphisms
of \( M \), and by \( \mathcal{H}(M)_0 \) its subgroup consisting of all homeomorphisms isotopic to the identity through an isotopy stabilizing outside a compact set. Note that \( \mathcal{H}(M)_0 \) is equal to the identity component in the compact-open topology whenever \( M \) is compact. This follows from the local contractibility of the \( \mathcal{H}(M) \) provided \( M \) is compact or equal to \( \mathbb{R}^n \) (cf. [3]).

Recall that in view of Alexander trick every compactly supported \( g \in \mathcal{H}(\mathbb{R}^n) \) is isotopic through \( \{g_t\} \) to \( \text{id} \), where

\[
g_t(x) = \begin{cases} tg(x/t) & \text{if } t > 0 \\ x & \text{if } t = 0. \end{cases}
\]

Thus the identity component of \( \mathcal{H}(\mathbb{R}^n) \) contains all compactly supported homeomorphisms. The same still holds if one considers \( \mathcal{H}(\mathbb{R}^n_+) \), where \( \mathbb{R}^n_+ = \{x_n \geq 0\} \) is the half-space. Note that the identity component of \( \mathcal{H}(\mathbb{R}^n) \) equals the group of all orientation preserving diffeomorphisms with possible exception \( n = 4 \) (cf. [9]).

Our purpose is to extend a theorem of Mather [11] from boundaryless manifolds to the general case. Next we formulate questions concerning homeomorphisms on a manifold with boundary, and we exhibit similarities and differences between homeomorphism and diffeomorphism groups. Specifically our result for homeomorphisms can be only partly extended to diffeomorphism groups (Theorem 4.1 and Proposition 4.2).

2. The main result. Let \( G \) be a topological group. The homology of \( G \) can be introduced in various equivalent ways (cf. [2, II]). Following Mather we introduce the homology of \( G \) as follows. Denote

\[
C_r(G) = \text{free abelian group on the set of all } r\text{-tuples } (g_1, \ldots, g_r),
\]

where \( g_i \in G \). Next introduce a differential \( \partial : C_r(G) \to C_{r-1}(G) \) by the formula

\[
\partial(g_1, \ldots, g_r) = (g_1^{-1}g_2, \ldots, g_1^{-1}g_r) + \sum_{i=1}^r (-1)^i (g_1, \ldots, \hat{g}_i, \ldots, g_r).
\]

One has the following well known property

\[
H_1(G) = G/[G, G].
\]

We also have
Proposition 2.1. Let \( g \in G \) and \( \alpha : G \to G \) be the inner automorphism of \( G \) given by \( \alpha(h) = ghg^{-1} \). Then \( \alpha_* : H_*(G) \to H_*(G) \) is the identity.

For the proof, see [2, section II,6].

Let \( G \) be \( \mathcal{H}(\mathbb{R}^n)_0 \) or \( \mathcal{H}(\mathbb{R}^n_+)_0 \), and \( \sigma = \sum k_j (g_{ij}, \ldots, g_{rj}) \), where \( k_j \in \mathbb{Z} \), be a chain from \( C_r(\mathcal{H}(\mathbb{R}^n)) \). We define the support of \( \sigma \) by

\[
\text{supp}(\sigma) := \bigcup_{i,j} \text{supp}(g_{ij}),
\]

where, for \( h \) being a homeomorphism, \( \text{supp}(h) \) equals the closure of the set \( \{ x : h(x) \neq x \} \). Thus \( \text{supp}(\sigma) \subset U \) iff \( \text{supp}(g_{ij}) \subset U \) for each \( i, j \).

In [11] Mather has proved that \( H_r(\mathcal{H}(\mathbb{R}^n)_0) = 0 \) for any positive integer \( r \). We wish to extend his argument for the homeomorphism group on the half-space \( \mathbb{R}^n_+ \).

Theorem 2.2. The homology groups \( H_r(\mathcal{H}(\mathbb{R}^n_+)_0) \) vanish for any \( r > 0 \).

Remark. Obviously \( H_0(\mathcal{H}(\mathbb{R}^n_+)_0) = \mathbb{Z} \).

From now on the symbol \( \mathcal{H}_U(\mathbb{R}^n_+)_0 \), where \( U \) is a closed interval, will stand for the totality of elements from \( \mathcal{H}(\mathbb{R}^n_+)_0 \) compactly supported in \( U \). By \( \iota : \mathcal{H}_U(\mathbb{R}^n_+)_0 \hookrightarrow \mathcal{H}(\mathbb{R}^n_+)_0 \) we denote the inclusion, and \( \iota_* : H_r(\mathcal{H}_U(\mathbb{R}^n_+)_0) \to H_r(\mathcal{H}(\mathbb{R}^n_+)_0) \) is the corresponding map on the homology level.

Lemma 2.3. \( \iota_* \) is an isomorphism.

Proof. (i) \( \iota_* \) is injective. Suppose \( \{ \sigma \} \in H_r(\mathcal{H}_U(\mathbb{R}^n_+)_0) \) and \( \iota_* \{ \sigma \} = 0 \). Then there is an \((r + 1)\)-cycle \( c \in C_{r+1}(\mathcal{H}(\mathbb{R}^n_+)_0) \) such that \( \partial c = \sigma \). Since \( \text{supp}(\sigma) \subset U \), it is easy to observe that there is a homeomorphism \( \phi \in \mathcal{H}(\mathbb{R}^n_+)_0 \) such that \( \phi = \text{id} \) in a neighbourhood of \( \text{supp}(\sigma) \) and \( \phi(\text{supp}(c)) \subset U \).

We have

\[
\partial(Ad_\phi c) = Ad_\phi (\partial c) = Ad_\phi \sigma = \sigma,
\]

and \( Ad_\phi c \in C_{r+1}(\mathcal{H}_U(\mathbb{R}^n_+)_0) \), as required.

(ii) \( \iota_* \) is surjective. Suppose \( \{ \sigma \} \in H_r(\mathcal{H}(\mathbb{R}^n_+)_0) \). The \( r \)-cycle \( \sigma \) having a compact support, one can find \( \phi \in \mathcal{H}(\mathbb{R}^n_+)_0 \) such that \( \text{supp}(Ad_\phi \sigma) = \phi(\text{supp}(\sigma)) \subset U \). In view of Proposition 2.1 we have \( \{ Ad_\phi \sigma \} = \{ \sigma \} \) so that the class \( \{ \sigma \} \) can be represented by a cycle supported in \( U \).

Proof of Theorem 2.2. Note that it suffices to prove the assertion for \( \mathcal{H}_U(\mathbb{R}^n_+)_0 \).

Let

\[
U = ([1, 2] \times [0, 1]^{n-2}) \times [0, 1] \subset \mathbb{R}^{n-1} \times \mathbb{R}_+ = \mathbb{R}^n_+.
\]

We can choose a homeomorphism \( \phi \in \mathcal{H}(\mathbb{R}^n_+)_0 \) verifying the condition

\[
\phi(x) = \frac{1}{3} x, \quad x \in B_+(0, 3),
\]

where \( B_+(0, 3) \) is the open ball of radius 3 in \( \mathbb{R}^n_+ \).
where $B_+(0, 3)$ is the half-ball with the center at 0 and radius 3. We set

$$U_j = \phi^j(U) = \left[ \frac{1}{3^j}, \frac{2}{3^j} \right] \times \left[ 0, \frac{1}{3^j} \right]^{n-1}$$

for $j = 1, 2, \ldots$. We have $U_j \cap U_k = \emptyset$ whenever $j \neq k$, and the sets $U_j$ "tend" to $0 \in \mathbb{R}_+^n$.

The proof proceeds by the induction on $r$. For $r = 0$ the assertion is vacuous. For the inductive step we may assume that $H_s(\mathcal{H}_U(\mathbb{R}_+^n)_0) = 0$ for $1 \leq s \leq r - 1$. By the Kunneth formula we then get

$$H_r(\mathcal{H}_U(\mathbb{R}_+^n)_0 \times \mathcal{H}_U(\mathbb{R}_+^n)_0) = H_r(\mathcal{H}_U(\mathbb{R}_+^n)_0) \oplus H_r(\mathcal{H}_U(\mathbb{R}_+^n)_0).$$

Fix arbitrarily $h_i \in \mathcal{H}_U(\mathbb{R}_+^n)_0, i = 1, \ldots, r$. We set for $\alpha = 0, 1$

$$\psi_\alpha(h_i)(x) = \phi^j h_i \phi^{-j}(x) \quad \text{for} \quad x \in U_j, \ j \geq \alpha,$$

$$= x \quad \text{otherwise}.$$

Note that $\psi_\alpha(h_i)$ is a well defined element of $\mathcal{H}(\mathbb{R}_+^n)_0$. Next by Proposition 2.1 we have

$$(*) \quad \{\psi_0(h_1, \ldots, h_r)\} = \{\psi_1(h_1, \ldots, h_r)\} \quad \text{in} \quad H_r(\mathcal{H}(\mathbb{R}_+^n)_0).$$

Indeed, we have $\psi_1(h_i) = \phi \psi_0(h_i) \phi^{-1}$.

Following [11] define $\eta : \mathcal{H}_U(\mathbb{R}_+^n)_0 \times \mathcal{H}_U(\mathbb{R}_+^n)_0 \rightarrow \mathcal{H}(\mathbb{R}_+^n)_0$ by $\eta(g_1, g_2) = g_1 \psi_1(g_2)$. As two homeomorphisms with disjoint supports commute and $supp(\psi_1(g_2)) \subset \bigcup_{j \geq 2} U_j$ we have $[g_1, \psi_1(g_2)] = id$. Hence $\eta$ is a group homomorphism, and

$$\psi_0 = \eta \circ \Delta,$$

where $\Delta$ is the diagonal map.

Now let $\{c\} \in H_r(\mathcal{H}_U(\mathbb{R}_+^n)_0)$. Then $\Delta_* \{c\} = \{c\} \oplus \{c\}$ by the above Kunneth formula. It follows by $(*)$ that

$$\psi_0 \star \{c\} = \eta \star \Delta \star \{c\} = \nu \star \{c\} + \psi_1 \star \{c\} = \nu \star \{c\} + \psi_0 \star \{c\}.$$ 

Thus $\nu \star \{c\} = 0$, and $\{c\} = 0$ by Lemma 2.3, as required.

**Corollary 2.4.** The groups $\mathcal{H}(\mathbb{R}_+^n)_0$ and $\mathcal{H}(\mathbb{R}_+^n)_0$ are perfect.

This is so since $H_1(\mathcal{H}(\mathbb{R}_+^n)_0) = 0$ and $H_1(\mathcal{H}(\mathbb{R}_+^n)_0) = 0$. 
Corollary 2.5. Let $M$ be a connected topological manifold. Then $\mathcal{H}(M)_0$ is a perfect group. Moreover if $\partial M = \emptyset$ then $\mathcal{H}(M)_0$ is simple.

Proof. Let $f \in \mathcal{H}(M)_0$. According to the partition property for homeomorphisms (cf. [3], Cor.1.3; actually this property holds for isotopies rather than for homeomorphisms) one has

$$f = f_s \circ \cdots \circ f_1,$$

where each $f_i$ is supported either in an open ball or in an open half-ball. Therefore one may assume that either $f \in \mathcal{H}(\mathbb{R}^n)_0$, or $f \in \mathcal{H}(\mathbb{R}_+^n)_0$, and $f$ is in the commutator subgroup by Theorem 2.1. Thus $\mathcal{H}(M)_0$ is perfect. The simplicity follows from the theorem of Epstein.

3. Two open problems. (1) $M$ being a manifold with boundary, let $\mathcal{H}(M, \partial)$ be the totality of homeomorphisms stabilizing the boundary $\partial$, and let $\mathcal{H}(M, \partial)_0$ be the subgroup of homeomorphisms isotopic to the identity through a compactly supported isotopy. Is $\mathcal{H}(M, \partial)_0$ perfect? Observe that the result would follow if

$$\iota_* : H_*(\mathcal{H}(M, \partial)_0) \rightarrow H_*(\mathcal{H}(M)_0)$$

were a monomorphism, where $\iota$ is the inclusion.

(2) Since results of Anderson [1] it is known that any stable homeomorphism is written as a product of commutators. A so-called Stable Homeomorphism Conjecture says that all orientation preserving homeomorphisms of $\mathbb{R}^n$ are stable. This conjecture has been proved except possibly $n = 4$ (see e.g.[9]). On the other hand, a main theorem of Kirby [9] states that the group of stable homeomorphisms of $\mathbb{R}^n$ is exactly the identity component of $\mathcal{H}(\mathbb{R}^n)$, that is, it consists of all homeomorphisms isotopic to the identity. These results and [5] allow McDuff [12] to prove that the identity component of $\mathcal{H}(\mathbb{R}^n)$ is perfect (it is obviously non-simple).

Our question is whether the result of McDuff can be extended onto open manifolds with boundary. Specifically, whether the identity component of $\mathcal{H}(\mathbb{R}^n_+)$ is perfect.

4. Diffeomorphism groups versus homeomorphism groups. In this section $M$ is an $n$-dimensional connected second countable $C^\infty$ smooth manifold with or without boundary. When comparing diffeomorphism groups with homeomorphism groups, first it should be noted that the identity component (in the $C^r$ compact-open topology) of the group of all $C^r$ diffeomorphisms, $Diff^r(M)_0$, where $1 \leq r \leq \infty$, consists of diffeomorphisms isotopic to the
identity. This again follows from the local contractibility of $\text{Diff}^r(M)$, and the proof is simpler than for $C^0$ because vector fields are in use. Next the partition property holds for the identity component of diffeomorphism group and the proof of it is much simpler than that for homeomorphisms, see e.g. [13], Lemma 3.1.

The Alexander trick is no longer true in case of diffeomorphisms, and this is the main reason that the simplicity and perfectness theorems are much more difficult in this case. Moreover we have that $\text{Diff}^\infty(\mathbb{R}^n)_c/\text{Diff}^\infty(\mathbb{R}^n)_0$ is equal to the Kervaire-Milnor group $\Gamma_{n+1}$ of homotopy of $(n+1)$-spheres (cf.[8], the subscript "c" indicates compactly supported diffeomorphisms). In particular, $\text{Diff}^\infty(\mathbb{R}^6)_c/\text{Diff}^\infty(\mathbb{R}^6)_0$ is equal to $\mathbb{Z}_{28}$.

Fukui in [7] has computed the first homologies of some diffeomorphism groups. Among others he proved that

$$H_1(\text{Diff}^\infty([0,1])_0) = \text{abelianization of } \text{Diff}^\infty([0,1])_0 = \mathbb{R}^2$$

and

$$H_1(\text{Diff}^\infty_r(\mathbb{R},0)_0) = \text{abelianization of } \text{Diff}^\infty_r(\mathbb{R},0)_0 = \mathbb{R}^{r+1},$$

where $\text{Diff}^\infty_r(\mathbb{R},0)_0$ is the group of all $C^\infty$ diffeomorphisms on $\mathbb{R}_+ r$-tangent to 0. These results imply that an analogue of Theorem 2.1 is not true in case of the above groups.

Furthermore, suppose the boundary $\partial \neq \emptyset$ and denote by $\text{Diff}^r_s(M, \partial)$ the group of all diffeomorphisms of class $C^r$ on $M$ which are $s$-tangent to the identity on the boundary, where $1 \leq s \leq r \leq \infty$. The condition of $s$-tangency means that the $s$-jets of a diffeomorphism and the identity are equal at any point of the boundary.

We have the following smooth counterpart of Theorem 2.2 due to Sergeraert [16] and Masson [10].

**Theorem 4.1.** The group $\text{Diff}^\infty(M, \partial)_0$ is perfect.

**Remark.** It is obviously non-simple as all its elements stabilizing near the boundary constitute a normal subgroup.

Next we have the following negative result which can be viewed as certain a generalization of [7].

**Proposition 4.2.** Let $1 \leq s < r \leq \infty$. Then $\text{Diff}^r_s(M, \partial)_0$ is not a perfect group.

**Proof.** The group $\text{Diff}^r_s(M, \partial)_0$ is not perfect whenever $1 \leq s < r$. In fact, let $(U, x_1, \ldots, x_n)$ be a local coordinate system at $p \in \partial M$ such that
$U = \{x_n \geq 0\}$. For any diffeomorphisms $u$ and $v$ we have the following composition formula for the differential of order $s + 1$

$$D^{s+1}(u \circ v) = (D^{s+1}u \circ v)(Dv \times \cdots \times Dv) + \sum C(i, j_1, \ldots, j_i)(D^iu \circ v)(D^{j_1}v \times \cdots \times D^{j_i}v) + (Du \circ v)(D^{s+1}v),$$

where $C(i, j_1, \ldots, j_i)$ is an integer independent of $n$, $1 < i < s + 1$, $j_l > 0$, and $j_1 + \cdots + j_i = s + 1$ (cf.[6]). Hence if $f, g \in Diff^s_0(M, \partial M)$ then

$$D^{s+1}(f \circ g)(p) = D^{s+1}f(p) + D^{s+1}g(p)$$

and

$$D^{s+1}f^{-1}(p) = -D^{s+1}f(p).$$

Therefore if we choose $h \in Diff^s_0(M, \partial)$ such that $D^{s+1}h(p) \neq 0$, the above equalities yield that $h$ cannot be in the commutator subgroup, as $D^{s+1}[f, g](p) = 0$.

References


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