ON BERTINI-TYPE THEOREM FOR WEAKLY-NORMAL COMPLEX ANALYTIC SETS

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1. Introduction. The aim of the paper is to present an elementary proof of the following Bertini-type theorem for weakly-normal complex analytic sets

THEOREM 1. If $X$ is a weakly-normal locally analytic subset of $\mathbb{C}^n$ then there exists a fat subset $M$ of the space of all affine hyperplanes in $\mathbb{C}^n$ such that for every $H \in M$ the intersection $X \cap H$ is again weakly-normal.

In [M, Corollary II.6] an analogous Bertini-type theorem for normal and reduced locally analytic sets in $\mathbb{C}^n$ is proved.

Proofs for normal and reduced complex analytic sets are very similar, a Bertini-type theorem is deduced from two facts: a Sard-type theorem and the openness condition for the given property. In [M1, Thm. 18.] Manaresi proved that this kind of arguments may be applied to any local property of complex spaces (not being only reduced and normal).

The Sard-type theorem for reduced and normal complex analytic sets follows from the homological characterization of those properties, openness conditions were proved by Bánica in ([Ba]). Using a similar characterization Manaresi (in [M, Thm. 1.12]) proved a Sard-type theorem for weakly-normal complex analytic sets. The proof of the openness condition for weakly-normal complex analytic sets turned out to be much more difficult. Bingener and Flenner in [B-F] proved the openness condition for local properties satisfying


Supported by KBN Grant P03A 061 08.
certain conditions and verified these conditions for weakly-normal complex analytic sets. The proof is very complicated, it is based on the method of Stein compacts and uses a lot of algebraic geometry.

Our proof of the Bertini-type theorem for weakly-normal complex analytic sets is completely elementary and in fact is based on a careful examination of the proof of the Sard-type theorem (given by Manaresi), especially the class of "wrong fibers".

2. Homological characterization of reduced and normal complex spaces. Let $X$ be a complex space. For $k = 0, 1, 2, \ldots$, denote

$$S_k(O_X) := \{ p \in X : \text{prof } O_{X, p} \leq k \} \subset X.$$  

In this situation $S_k(O_X)$ is an analytic subset of $X$, for any $k$ ([S-T]). These "singular subsets" $S_k$ give a lot of information about the space $X$. In particular, we have the following characterization of reduced and normal spaces

**Lemma 1** [S-T]. $X$ is reduced iff

$$\dim(Sing X \cap S_k(O_X)) \leq k - 1,$$

for any $k \geq 0$.

**Lemma 2** [M]. $X$ is normal iff

$$\dim(Sing X \cap S_k(O_X)) \leq k - 2,$$

for any $k \geq 1$.

3. Weakly-normal complex spaces. A complex space $X$ is said to be weakly-normal (or maximal) if every $c$-holomorphic function on $X$ is holomorphic. Detailed information on weakly normal complex spaces may be found in [F] (see also [A-N]), we shall only give the following characterization here.

Let $\pi: \bar{X} \to X$ be a normalization of $X$. Consider the reduction of the fiber product $R := (\bar{X} \times_X \bar{X})_{\text{red}}$, $\pi': R \to X$.

Denote by $g_1, g_2: (\bar{X} \times_X \bar{X})_{\text{red}} \to \bar{X}$ the mappings induced by the projections $p_1, p_2: \bar{X} \times_X \bar{X} \to \bar{X}$. Let $O_X$ and $O_{\bar{X}}$ denote the structure sheaves of $X$ and $\bar{X}$.

**Lemma 3**, [M, (0.4.1)]. The complex space $(X, O_X)$ is weakly-normal iff the sequence

$$0 \to O_X \xrightarrow{\pi_*} \pi_*O_{\bar{X}} \xrightarrow{(g_1 - g_2)^*} \pi'_*O_R$$

is exact.
4. Bertini-type theorems for reduced and normal complex analytic sets. Now, let \( f \in \mathcal{O}_X(X) \) be a holomorphic function on a complex space \( X \). We shall denote \( X_f := f^{-1}(0) \) and \( \mathcal{O}_{X_f} := \mathcal{O}_X/(f \cdot \mathcal{O}_X) \). In this situation \( (X_f, \mathcal{O}_{X_f}) \) is a complex space. We shall denote it by \( X_f \).

Following the ideas of [M] we can give a proof of a version of a Bertini-type theorem for reduced and normal complex analytic sets.

**Lemma 4.** Let \( X \) be a reduced (resp. normal) complex space. If \( f \in \mathcal{O}_X(X) \) is such that

1. \( \text{Sing} X_f \subset \text{Sing} X \)

and

2. \( X_f \) does not contain any irreducible component of sets \( \text{Sing} X \cap S_k(\mathcal{O}_X) \)

then the complex space \( X_f \) is reduced (resp. normal).

**Proof.** We shall prove the lemma for a reduced complex space. By the assumption (2) the germ of \( f \) is not a zero divisor in any local ring \( \mathcal{O}_{X,x} \), so for every \( x \in X_f \) we have \( \text{prof}(\mathcal{O}_{X,x}) = \text{prof}(\mathcal{O}_{X_f,x}) + 1 \) and consequently \( S_k(\mathcal{O}_{X_f,x}) \subset S_{k+1}(\mathcal{O}_X) \) so by (1), we have \( \text{Sing} X_f \cap S_k(\mathcal{O}_{X_f}) \subset S_{k+1}(\mathcal{O}_X) \cap X_f \).

By assumption (2) and Lemma 1 this gives \( \dim(\text{Sing} X_f \cap S_k(\mathcal{O}_{X_f})) \leq \dim(\text{Sing} X \cap S_{k+1}(\mathcal{O}_X)) - 1 \leq k + 1 - 1 - 1 = k - 1 \). By Lemma 1, this completes the proof.

The proof for a normal space is similar (we use Lemma 2 instead of Lemma 1).

From this lemma, there easily follows Bertini–type Theorem for normal and reduced complex analytic sets.

**Theorem 2.** [M, Cor. II.6]. If \( X \) is a normal (resp. reduced) locally analytic subset of \( \mathbb{C}^n \) then there exists a fat subset \( M \) of the space of all affine hyperplanes in \( \mathbb{C}^n \) such that for every \( H \in M \) the intersection \( X \cap H \) is again normal (resp. reduced).

5. Bertini-type theorem for weakly-normal analytic sets. We shall preserve the notation introduced in previous sections. Moreover, for simplicity we shall denote \( \tilde{X}_f := \tilde{X}_{f_0 \pi} \), \( R_f := R_{f_0 \pi'} \).

**Lemma 5.** Let \((X, \mathcal{O}_X)\) be a complex space, and let

\[
0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}
\]

be an exact sequence of coherent analytic sheaves. Then there exists a sequence of analytic subsets \( \{X_i\}_{i \in I} \) of \( X \) such that for any holomorphic function \( f \in \mathcal{O}_X(X) \) satisfying condition:

\( X_f \) does not contain any of the sets \( X_i \).
the sequence
\[ 0 \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \xrightarrow{\alpha} \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \xrightarrow{\beta} \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \]
is exact.

**Proof.** Take as \( X_i \) all analytic varieties associated with the sheaves \( \mathcal{G}/\alpha(\mathcal{F}), \mathcal{H}/\text{Im}\beta ([S]) \) and irreducible components of space \( X \). Then apply the proof of [M (1.8)].

**Lemma 6.** Let \( X \) be a weakly-normal complex space. There exists a sequence of analytic subsets \( \{X_i\} \) of \( X \) such that if \( f \in \mathcal{O}_X(X) \) is such that

1. \( f \) is not zero on any of sets \( X_i \),
2. \( \text{Sing } R_f \subset \text{Sing } R \),
3. \( \text{Sing } \tilde{X}_f \subset \text{Sing } \tilde{X} \),

then the complex space \((X_f, \mathcal{O}_{X_f})\) is weakly-normal.

**Proof.** Let us take the exact sequence
\[ 0 \rightarrow \mathcal{O}_X \xrightarrow{\pi_*} \pi_*\mathcal{O}_{\tilde{X}} \xrightarrow{(g_1-g_2)_*} \pi'_*\mathcal{O}_R. \]

There exists a sequence of analytic subsets \( \{X_i\} \) of \( X \) such that for any holomorphic function \( f \in \mathcal{O}_X(X) \) which is not zero on any of sets \( X_i \) we have

(a) the sequence
\[ 0 \rightarrow \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \xrightarrow{\pi_*} \pi_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \xrightarrow{(g_1-g_2)_*} \pi'_*\mathcal{O}_R \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} \]
is exact,

(b) the space \((\tilde{X}_f, \mathcal{O}_{\tilde{X}_f})\) is normal,

(c) the space \((R_f, \mathcal{O}_{R_f})\) is reduced,

(d) the set \( X_f \) contains no irreducible component of \( X \) and \( \text{Sing } X \).

Now, we have
\[ \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} = \mathcal{O}_{X_f}, \]
\[ \pi_*\mathcal{O}_{\tilde{X}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} = \pi_*\mathcal{O}_{\tilde{X}_f}, \]
\[ \pi'_*\mathcal{O}_R \otimes_{\mathcal{O}_X} \mathcal{O}_{X_f} = \pi'_*\mathcal{O}_{R_f}. \]

In this situation, by (b) and (d)
\[ \pi|_{\tilde{X}_f} : \tilde{X}_f \rightarrow X_f \]
is a normalization.

Using the universal property of the fiber product and (c) we get

\[(\tilde{X}_f \times_{X_f} \tilde{X}_f)_{\text{red}} = R_f.\]

Consequently, we can write the exact sequence (a) in the following form

\[0 \rightarrow \mathcal{O}_{X_f} \rightarrow \mathcal{O}_{\tilde{X}_f} \rightarrow (\mathcal{O}_{\tilde{X}_f \times_{X_f} \tilde{X}_f})_{\text{red}} \]

which completes the proof.

**Proof of Theorem 1.** Let $X$ be a weakly-normal analytic subset of $\mathbb{C}^n$. From the Sard theorem it follows that there exists a fat family $M_1$ of hyperplanes in $\mathbb{C}^n$ such that for any $H \in M_1$ we have

\[\text{Sing } (X \cap H) \subset \text{Sing } X \cap H.\]

Now, let \(\{X_i\}\) be a sequence of analytic subsets of $X$ satisfying the assertions of Lemma 6. The family $M_2$ of those affine hyperplanes in $\mathbb{C}^n$ which do not contain any $X_i$ is fat.

Then the intersection $M := M_1 \cap M_2$ is also fat. Take any $H \in M$ and let $f$ be an equation of $H$. Since the mappings $\pi$ and $\pi'$ are biholomorphisms outside the singular locus $\text{Sing } X$ of $X$, we have $\text{Sing } \tilde{X}_f \subset \text{Sing } \tilde{X}$ and $\text{Sing } R_f \subset \text{Sing } R$. Therefore, by Lemma 6, the set $X \cap H = X_f$ is weakly-normal. This proves the Theorem.

**Acknowledgments.** A part of this research was done during the authors stay at the Mathematical Institute of the Bologna University supported by Consiglio Nazionale delle Ricerche (grant number 211.01.27).

The author would like to thank the staff of the Institute for their hospitality and Prof. Manaresi numerous fruitful discussions.

**References**


Received May 25, 1995

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