GAUSSIAN MEASURE – PRESERVING LINEAR TRANSFORMATIONS

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ABSTRACT. Let $T$ be a linear continuous transformation in a separable Banach space. It is proved that if $T$ has an invariant mixing measure with Gaussian covariance, then $T$ also has an invariant mixing Gaussian measure. This theorem is applied to contradictions in Hilbert spaces and weighted shift transformations in $l_p$ and $c_0$.

1. Introduction

In this paper we study the problem of existence of invariant measures for linear transformations in Banach spaces. This problem was investigated for transformations generated by partial differential equations in the papers of Lasota [8], Brunovsky and Komornik [1] and Rudnicki [9,10,11]. There seem to be two methods of construction of invariant measures for such equations. The first, presented in the paper of Lasota [8], is based on the Kriloff–Bogoluboff theorem. This method can also be used for nonlinear transformations, but it has one disadvantage, since the Kriloff–Bogoluboff theorem guarantees only the existence of an invariant measure and its ergodicity. In the second method [1,9,10,11] an invariant measure is induced by a Gaussian stochastic process. This approach allows us to prove such properties of the invariant measure as mixing, exactness and positivity on open sets.

The second method can be used to examine other linear transformations. In this general case we can give necessary and sufficient conditions for the existence of an invariant mixing measure in the class of Gaussian measures. Our aim is to demonstrate the essential unity of both methods. Precisely, we shall prove that under some assumptions, the existence of an invariant probabilistic measure implies the existence of an invariant Gaussian measure. This Gaussian measure has at least the same ergodic and analytic properties.
as the previous one. The general approach is illustrated by analysis of two
examples: contractions in Hilbert spaces and weighted shift transformations
on the spaces $l_p$ and $c_0$. In the first example we use some properties of linear
measure preserving transformations given by Flytcen [5].

2. Preliminaries

We introduce some definitions and results concerning probabilistic measures
in Banach spaces [2,3,7].

Let $X$ be a separable Banach space and $X^*$ the dual space. By $M(X)$
we denote the set of probabilistic measures defined on the $\sigma$-algebra $B(X)$ of
Borel subsets of $X$. A measure $\mu \in M(X)$ is called a weakly second-order
measure if $\int x^*^2(x)\mu(dx) < \infty$ for all $x^* \in X$. Every weakly second-order
measure $\mu$ has a mean value $m_\mu \in X$ and a covariance $R_\mu : X^* \rightarrow X$
which are uniquely determine by the formulae

$$x^*(m_\mu) = \int_X x^*(x)\mu(dx),$$
$$y^*(R_\mu x^*) = \int_X x^*(x)y^*(x)\mu(dx) - x^*(m_\mu)y^*(m_\mu),$$

$x^*, y^* \in X^*$. If $m_\mu = 0$ then $\mu$ is called a centred measure. Every covariance
$R$ is symmetric, i.e., $x^*_2(Rx^*_1) = x^*_1(Rx^*_2)$ for $x^*_1, x^*_2 \in X^*$, and positive, i.

$$x^*(R x^*) \geq 0 \text{ for } x^* \in X^*. \text{ A measure } \mu \in M(X) \text{ is called a strongly}
\text{second-order measure if } \int_X ||x||^2\mu(dx) < \infty.$$

A measure $\mu \in M(X)$ is called Gaussian if all the functionals $x^* \in X^*$,
regarded as random variables on $(X, \mu)$ are (possibly degenerate) Gaussian
random variables. A centred Gaussian measure $\mu$ is uniquely determined
by its covariance. We say that a measure $\mu \in M(X)$ has Gaussian covariance
$R_\mu$ if $R_\mu$ coincides with a covariance of some Gaussian measure. The class
of measures having Gaussian covariances is relatively large. For instance, if
$X$ is a separable Hilbert space then any strongly second-order measure has a
Gaussian covariance.

We also need a description of Gaussian covariances in the spaces $l_p$ and $c_0$.
A positive, symmetric operator $R : X^* \rightarrow X$ is a covariance of a Gaussian
measure in $l_p$, $1 \leq p < \infty$, iff

$$\sum_k (\epsilon_k^*(R \epsilon_k^*))^{p/2} < \infty$$

where $(\epsilon_k^*)_{k \in \mathbb{N}}$ is the natural base in $l_p^*$. The problem of general description
of Gaussian covariance in the space $c_0$ is unsolved. In the special case when
$R$ is of the form $Rx^* = \sum \lambda_k x^*(e_k)e_k$, then $R$ is a Gaussian covariance in $c_0$ iff $\sum \exp(-\frac{\alpha}{\lambda_k}) < \infty$ for every $\alpha > 0$, [2,4].

Let $\mu \in M(X)$. The smallest closed subset $F$ of $X$ such that $\mu(F) = 1$ is called the support of $\mu$ and it will be denoted by $\text{supp} \mu$. The smallest closed linear subspace $E$ of $X$ such that $\mu(E) = 1$ is called the linear support of $\mu$ and will be denote by $V_\mu$. It is easy to verify that if $\mu$ is a centred measure and has the covariance $R_\mu$ than $V_\mu = \text{cl} R_\mu(X^*)$ (cl=closure).

3. Main theorem

Before formulating the main result we recall that a measurable transformation $T : X \to X$ is mixing with respect to a probabilistic measure $\mu$ if $\lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B)$ for all $A, B \in \mathcal{B}(X)$.

**Theorem 1.** Let $T : X \to X$ be a continuous linear operator defined on a separable Banach space $X$ and let $\mu$ be a probabilistic measure on $X$ invariant under $T$. Assume that $\mu$ has a Gaussian covariance $R$. Then the Gaussian centred measure $m$ having the covariance $R$ is also invariant under $T$ and $\text{supp} \ m = V_\mu$. Moreover, if $T$ is mixing with respect to $\mu$, then $T$ is also mixing with respect to $m$.

**Proof.** The proof splits into four parts.
1. First observe that if $\mu$ is invariant under $T$ then $TRT^* = R$, where $T^*$ denotes the conjugate operator of $T$. Indeed, put $\nu(A) = \frac{1}{2}(\mu(A) + \mu(-A))$. Then the measure $\nu$ is centred and $R$ is also the covariance of $\nu$. Since the measure $\nu$ is invariant under $T$ we have

$$y^*(Rx^*) = \int y^*(x)x^*(x)\nu(dx) = \int y^*(Tx)x^*(Tx)\nu(dx)$$

$$= (Ty^*)(R(Tx^*)) = y^*(TRT^*(x^*))$$

for every $x^*, y^* \in X^*$. This implies that $TRT^* = R$. Now, assume that $m$ is a centered Gaussian measure and $R$ is its covariance. Let $\overline{m}(A) = m(T^{-1}(A))$ for $A \in \mathcal{B}(X)$. Then $\overline{m}$ is also a Gaussian measure and

$$\int y^*(x)x^*(x)\overline{m}(dx) = \int y^*(Tx)x^*(Tx)m(dx).$$

The last formula implies that $TRT^*$ is a covariance operator of $\overline{m}$. Thus the measures $m$ and $\overline{m}$ have the same covariance and consequently $m(A) = m(T^{-1}(A))$ for $A \in \mathcal{B}(X)$. 

2. We verify that \( \text{supp}\ m = V_\mu \). From the definition of \( \nu \) we have \( V_\nu = V_\mu \). Since the support of any Gaussian measure is a linear subspace we have \( V_m = \text{supp}\ m \). The measures \( m \) and \( \nu \) have the same covariance. This implies that \( V_m = \text{cl}(R(X^*)) = V_\nu \) and consequently \( \text{supp}\ m = V_\nu = V_\mu \).

3. Now we show that if \( \mu \) is mixing then

\[
\lim_{n \to \infty} y^*(RT^n x^*) = 0 \text{ for } x^*, y^* \in X^*.
\]

Since the measure \( \mu \) is invariant, \( T^n m_\mu = m_\mu \) for every \( n \geq 0 \). Hence

\[
y^*(RT^n x^*) = \int_X (T^n x^*)(x)y^*(x)d\mu(x) - (T^n x^*)(m_\mu)y^*(m_\mu)
\]

\[
= \int_X x^*(T^n x)y^*(x)d\mu(x) - \int_X x^*(x)d\mu \int_X y^*(x)d\mu.
\]

Since \( \mu \) is a weakly second-order measure, \( x^*, y^* \in L_2(\mu) \). Mixing implies that the last term converges to zero as \( n \to \infty \).

4. Finally, we verify that if \( m \) is a centred Gaussian measure invariant under \( T \) and its covariance satisfies (3.1), then \( (T, m) \) is mixing. Let \( F \) be the family of all cylinder sets of the form

\[
\{x \in X: (x^*_1(x), \ldots, x^*_k(x)) \in E\},
\]

where \( k \) is an arbitrary integer, \( E \) is a Borel subset of \( R^k \) and \( x^*_1, \ldots, x^*_k \) are elements of \( X^* \). Additionally, we assume that the join distribution of \( (x^*_1, \ldots, x^*_k) \) is non-degenerate. The family \( F \) is an algebra which generates the \( \sigma \)-algebra \( B(X) \). Thus for every \( A \in B(X) \) and \( \varepsilon > 0 \) there exists \( A' \in F \) such that \( m(A \triangle A') < \varepsilon \). Due to the fact that \( m \) is invariant under \( T \), it is sufficient to verify that \( \lim_n m(A \cap T^{-n}(B)) = m(A)m(B) \) for all \( A, B \in F \).

Let

\[
A = \{x \in X: (x^*_1(x), \ldots, x^*_k(x)) \in E\},
\]

\[
B = \{x \in X: (y^*_1(x), \ldots, y^*_p(x)) \in F\},
\]

where \( E \in B(R^k) \), \( F \in B(R^p) \) and \( (x^*_1, \ldots, x^*_k) \) and \( (y^*_1, \ldots, y^*_p) \) are non-degenerate sequences from \( X^* \). Then

\[
T^{-n}(A) \cap B
\]

\[
= \{x \in X: (T^n x_1(x), \ldots, T^n x_k(x), y^*_1(x), \ldots, y^*_p(x)) \in E \times F\}.
\]
Denote respectively by \( f(u), g(v) \) and \( h_n(u, v) \) the densities of Gaussian random vectors

\[
(x_1^*, \ldots, x_k^*), (y_1^*, \ldots, y_p^*) \text{ and } (T^{n^*} x_1, \ldots, T^{n^*} x_k, y_1^*, \ldots, y_p^*).
\]

From (3.1) it follows that for every \( x^*, y^* \in X^* \) we have

\[
\lim_{n \to \infty} \int_X y^*(x)(T^{n^*} x^*)(x) \mu(dx) = \lim_{n \to \infty} y^*(RT^{n^*} x^*) = 0.
\]

Since \( f, g \) and \( h_n \) are densities of centred Gaussian random vectors, the last condition implies that \( h_n(u, v) \) converges to \( f(u)g(v) \) in \( L_1(R^{k+p}) \) as \( n \to \infty \). Hence

\[
\lim_{n \to \infty} m(T^{-n}(A) \cap B) = \lim_{n \to \infty} \int_{E \times F} h_n(u, v) \, du \, dv = \int_F \int_E f(u)g(v) \, du \, dv = m(A)m(B)
\]

which completes the proof.

4. Contractions in Hilbert spaces

The following proposition shows that if \( T \) is a contraction in a Hilbert space then we may eliminate one assumption of Theorem 1, namely, that \( \mu \) has Gaussian covariance.

**Proposition 1.** Let \( T : H \to H \) be a linear contraction in a real separable Hilbert space \( H \). Assume that there exists a probabilistic measure \( \mu \) invariant under \( T \). Then there exists an invariant Gaussian measure \( m \) such that \( \text{supp } m = V_\mu \).

**Proof.** Let \( r \) be a positive integer such that \( \mu(K_r) > 0 \). Define a new probabilistic measure \( \nu \) by

\[
\nu(A) = \sum_{n=r}^{\infty} \frac{\mu(A \cap K_n)}{\mu(K_n)} 2^{-n+r-1},
\]

where \( K_n = \{ x \in H : \| x \| \leq n \} \). Since \( K_n \subset T^{-1}(K_n) \) and \( \mu \) is invariant under \( T \) we have \( \mu(T^{-1}(K_n) \setminus K_n) = 0 \). This implies that \( \nu \) is invariant.
under \( T \). Moreover \( \text{supp} \nu = \text{supp} \mu \). Notice that the measure \( \nu \) has a finite second-order. Indeed,

\[
\int_X \|x\|^2 \nu(dx) \leq r^2 \nu(K_r) + \sum_{n=r+1}^{\infty} n^2 \nu(K_n \setminus K_{n-1})
\leq r^2 \nu(K_r) + \sum_{n=r+1}^{\infty} 2^{-n+r} n^2 < \infty.
\]

Hence the covariance of \( \nu \) is Gaussian. According to Theorem 1 the operator \( T \) has an invariant Gaussian measure \( m \) such that \( \text{supp} \ m = V_\nu = V_\mu \).

Now we apply Proposition 1 to contractions in a complex Hilbert space \( H \). Such transformations were considered in [5,6]. Every complex Hilbert space is also a real Hilbert space if we replace the complex inner product \( \langle x|y \rangle \) by the real inner product \( \langle x|y \rangle = \text{Re}(x|y) \). This identification preserves the norm in \( H \) and a contraction in \( H \) is also a contraction in the new real Hilbert space. From Proposition 1 and Theorem 1 [5] it follows immediately

**Corollary 1.** A contraction \( T \) in a complex separable Hilbert space \( H \) has an invariant Gaussian measure \( \mu \) whose support spans \( H \) iff \( H \) is spanned by eigenvectors of \( T \) having unimodular (norm 1) eigenvalues.

**Remark 1.** Corollary 1 remains true if \( T \) is a contraction in a real separable Hilbert space \( H \). In fact, let \( H^- = H \oplus iH \) be the complexification of \( H \). Then \( T^- (x + iy) = T(x) + iT(y) \) is a contraction in \( H^- \) and there is the following relation between measures invariant under \( T \) and \( T^- \). If \( \mu \) is a Gaussian measure invariant under \( T \), then \( \mu \times \mu \) is a Gaussian measure invariant under \( T^- \). Inversely, if \( \mu^- \) is a Gaussian measure invariant under \( T^- \), then \( \mu(A) = \mu^- (A \oplus iH) \) is a Gaussian measure invariant under \( T \).

**5. Weighted shift transformations**

In this section we apply Theorem 1 to operators \( T \) given by the formula

\[(5.1) \quad (Tx)_k = \left( \frac{\lambda_k}{\lambda_{k+1}} \right) x_{k+1}, \quad k \in \mathbb{N}.
\]

The operators \( T \) will be defined in the Banach spaces of real sequences \( x = (x_1, x_2, \ldots) \) namely, on the space \( l_p \), \( 1 \leq p \leq \infty \), with the norm \( \|x\| = (\sum_k x_k^p)^{\frac{1}{p}} \) or on the space \( c_0 \) of all sequences convergent to 0 with the norm \( \|x\| = \sup_n |x_n| \).
Example 1. Let \((\lambda_k) \in l_p\) and satisfies the following condition
(i) there exists \(c > 0\) such that \(0 < |\lambda_k| < c|\lambda_{k+1}|\) for \(k \in \mathbb{N}\).

Then \(T\) is a continuous linear operator on \(X = l_p\). We verify that there exists a Gaussian measure \(m\) such that \(\text{supp } m = X\), \(m\) is invariant under \(T\), and \(T\) is mixing. In order to prove this we define the set \(Y\) by

\[ Y = \{(x) \in l_p : |x_k| = |\lambda_k| \quad \text{for } k \in \mathbb{N} \} \]

Let \(\mu^- = \prod_{k=1}^{\infty} \mu_k\) be a product measure on \(Y\), where \(\mu_k\{\lambda_k\} = \mu_k\{-\lambda_k\} = \frac{1}{2}\).

The set \(Y\) is invariant under \(T\) and the dynamical system \((Y, B(Y), \mu^-, T)\) is isomorphic to the standard Bernoulli shift. This implies that \(\mu^-\) is invariant under \(T\) and mixing. Let \(\mu\) be the extension of \(\mu^-\) on the whole space \(X\), i.e., \(\mu(A) = \mu^-(A \cap Y)\) for \(A \in B(X)\). The measure \(\mu\) has the covariance \(R_x^* = \sum \lambda_k^2 x^*(e_k)e_k\), where \((e_k)_{k \in \mathbb{N}}\) is the natural basis in \(l_p\). According to (2.1) \(R\) is the covariance of a Gaussian measure \(m\). Moreover, \(V_{\mu} = X\). Thus, Theorem 1 implies that \(m\) has all required properties.

Example 2. Let \((\lambda_k) \in c_0\) and satisfies (i). Then \(T\) is a continuous linear operator on \(X = c_0\) preserving the set

\[ Y = \{(x) \in c_0 : |x_k| = |\lambda_k| \quad k \in \mathbb{N} \} \]

Analogously as in Example 1, we can construct a probabilistic measure \(\mu\) on \(X\) invariant under \(T\) such that \(V_{\mu} = X\). With this measure the transformation \(T\) becomes mixing. The measure \(\mu\) has the covariance \(R_x^* = \sum \lambda_k^2 x^*(e_k)e_k\).

The operator \(R\) is a covariance of any Gaussian measure on the space \(c_0\) iff

\[ \sum \exp(-\alpha \lambda_k^{-2}) < \infty \quad \text{for every } \alpha > 0. \]

Thus if the sequence \((\lambda_k)\) satisfies (5.2) then \(T\) has an invariant Gaussian measure such that \(\text{supp } m = X\) and \(T\) is mixing.

Remark 2. Example 2 also shows that the covariance of an invariant measure is not generally a Gaussian covariance, e.g. if \(\lambda_k = \ln^{-\frac{1}{2}}(k + 1)\) then \(R\) is not a Gaussian covariance. But even in this case it is possible to construct a Gaussian measure \(m\) invariant under \(T\) such that \(\text{supp } m = X\). However, this construction is technically difficult and we omit the details here. This leads to the following open questions:

Let \(T\) be a linear continuous transformation on a separable Banach space \(X\) with an invariant probabilistic measure whose support spans \(X\).

1. Does there exists a non-trivial \((m\{0\} = 0)\) Gaussian measure invariant under \(T\)?
2. Does there exist an invariant Gaussian measure \( m \) such that \( \text{supp } m = X \)?

Remark 3. The condition \( TRT^* = R \) may not imply the existence of an invariant probabilistic measure if we omit the assumption that \( R \) is a Gaussian covariance. Consider the transformation \((Tx)_k = x_{k+1}, k \geq 1\), on the space \( l_2 \). It is easy to verify that \( T \) does not admit non-trivial invariant probabilistic measures. On the other hand the operator \( Rx^* = \sum x^*(e_k)e_k \) is a covariance of the measure \( \mu \) in \( l_2 \) concentrated in the points \( \pm 2^{\frac{1}{2}} e_k, k \in \mathbb{N}, \mu\{2^{\frac{1}{2}} e_k\} = \mu\{-2^{\frac{1}{2}} e_k\} = \frac{1}{2^{k+1}} \) and \( R \) satisfies \( TRT^* = R \).

References


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