ON SMOOTH DEPENDENCE OF SOLUTIONS
OF PARABOLIC EQUATIONS ON COEFFICIENTS

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In this paper we deal with the dependence of the solution of the parabolic equation (1) – (3) on the coefficients \( a_i \) (\( i = 1, \ldots, n \)), \( b \) on functions \( u_0, f \), in the case when \( \Omega \) is an open subset of \( \mathbb{R}^n \).

\[
\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) + \sum_{i=1}^{n} a_i(t,x) \frac{\partial u}{\partial x_i}(t,x) + b(t,x) \cdot u(t,x) = f(t,x), \quad 0 < t < T, x \in \Omega
\]

(2) \[ u(0,x) = u_0(x), \quad x \in \Omega \]

(3) \[ u(t,x) = 0, \quad 0 < t < T, \quad x \in \partial \Omega \]

We assume that

\[
a_i, b \in L^\infty([0,T] \times \Omega) = L^\infty(\Omega_T), \quad T < \infty, \quad i = 1, \ldots, n.
\]

We shall prove that the function \((a_i, b) \mapsto u\) is analytic (in appropriate spaces). To show this we will use the notations and theorems from [3].

Let us denote

\[
H = L^2(\Omega), \quad V = H_0^1(\Omega)
\]

We identify \( H \) with its antidual \( H' \) (that is \( H = H' \)) and following [3] we have:

\[
V' = H^{-1}(\Omega), \quad V \subset H \subset V',
\]

where the imbeddings are continous and dense.
By \((,\,,\,>)\), \(<\,,\,>\) and \(||\,,\,||\,\>\) we denote the scalar products and norms in \(V\) and \(H\), respectively.

For any \(t \in [0, T]\) we define the following form:

\[
a(t, \cdot, \cdot) : V \times V \ni (u, v) \mapsto \sum_{i=1}^{n} \int_{\Omega} a_i(t, x) D_i u(x) \cdot D_i v(x) dx \]

\[
+ \int_{\Omega} b(t, x) u(x) v(x) dx + \sum_{i=1}^{n} \int_{\Omega} D_i u(x) D_i v(x) dx
\]

(7)

Then we get that for any \(u, v \in V\) the function \(t \mapsto a(t, u, v)\) is measurable and from the Schwartz inequality it follows that

\[
\exists c > 0 \ | a(t, u, v) | \leq c |u| \cdot |v|, \quad \forall u, v \in V, t \in [0, T]
\]

(8)

We denote \(c_1 := \max \{ |a_i|_{L^{\infty}(\Omega_T)} \}, c_0 := |b|_{L^{\infty}(\Omega_T)}\). Since for any \(\varepsilon > 0\) we have

\[
\left| \int_{\Omega} a_i(t, x) D_i u(x) \overline{v(x)} dx \right| \leq \frac{1}{2} c_1 \varepsilon |D_i u|^2 + \frac{1}{2} |u|^2,
\]

\[
\left| \int_{\Omega} b(t, x) u(x) \overline{v(x)} dx \right| \leq c_0 |u|^2
\]

we have

\[
\text{Re } a(t, u, u) \geq \sum_{i=1}^{n} \left[ |D_i u|^2 - \frac{\varepsilon}{2} c_1 |D_i u|^2 \right] - \frac{n}{2 \varepsilon} c_1 |u|^2 - c_0 |u|^2
\]

\[
= \left( \sum_{i=1}^{n} |D_i u|^2 \right) \cdot \frac{2 - \varepsilon c_1}{2} - \left( c_0 + \frac{nc_1}{2 \varepsilon} \right) |u|^2
\]

Thus, if \(\varepsilon \leq c_1^{-1}\) and \(\lambda \geq \lambda_0 := c_0 + \frac{nc_1}{2 \varepsilon} + 1\) then

\[
\text{Re } a(t, u, u) + \lambda |u|^2 \geq \sum_{i=1}^{n} |D_i|^2 + |u|^2 = ||u||^2
\]

(9)

That means that the form \(a(t, \cdot, \cdot)\) is coercive with respect to \(V\). Next we introduce the following spaces:

\[
V := L^2(O, T, V), \quad \mathcal{H} := L^2(O, T, H)
\]

(10)
By $\mathcal{V}'$ we denote the antidual space to $\mathcal{V}$ and we have

\begin{equation}
\mathcal{V}' = L^2(\mathcal{O}, T, \mathcal{V}').
\end{equation}

Since (8) the form $a(t, \cdot, \cdot)$ is continous on $V \times V$ (for $t$ fixed), so there exists the unique $A(t) \in \mathcal{L}(V, V')$ such that

\begin{equation}
(A(t)u)(v) = a(t, u, v), \quad \forall u, v \in V
\end{equation}

We shall treat the following equation

\begin{equation}
\frac{du}{dt} + A(t)u(t) = f(t), \quad 0 < t < T
\end{equation}

which is a functional version of the equation (1). If $f \in L^2(\mathcal{O}, T, \mathcal{V}') = \mathcal{V}'$ then $u$ is a solution of (13) iff $u \in \mathcal{V} = L^2(\mathcal{O}, T, V)$, $u' = \frac{du}{dt}$ and the equation (13) is satisfied in the sense of distributions on $(0, T)$ with values in $\mathcal{V}'$. We shall recall some facts concerning the problem of existence and uniqueness of solutions of the equation (13), together with the initial condition $u(0) \in u_0$. Let us denote

\begin{equation}
Y := \{u \in \mathcal{V} : u' \in \mathcal{V}'\}, \quad |u|_Y^2 := |u|_V^2 + |u'|_{\mathcal{V}'}^2.
\end{equation}

$Y$ is a Hilbert space and because of the Trace Theorem (see [3] Th. 3.1) if $u \in Y$ then $u \in \mathcal{C}([0, T], H)$ (after possible modification on a set of measure 0) and the imbedding $Y \hookrightarrow \mathcal{C}([0, T], H)$ is continuous. If $X := \{u \in Y : u(0) = 0\}$ then $X$ is a closed subspace of $Y$. We have the following

**Lemma 1.** For any $\lambda \in \mathbb{R}$ the transformation $E_\lambda : u \mapsto \{t \mapsto e^{\lambda t}u(t)\}$ is an isomorphism from $X$ onto $X$, from $Y$ onto $Y$, from $\mathcal{V}$ onto $\mathcal{V}$ and from $\mathcal{V}'$ onto $\mathcal{V}'$.

**Proof of Lemma 1:** We shall prove that $E_\lambda$ is an isomorphism from $\mathcal{V}$ onto $\mathcal{V}$, (in the same way can prove the remaining part of Lemma 1). If $u \in \mathcal{V}$ then the function $\{t \mapsto e^{\lambda t}u(t)\} \in \mathcal{V}$ and $|e^{\lambda t}u(t)|_\mathcal{V} \leq e^{\lambda|T|} |u|_\mathcal{V}$ (because $T < \infty$). But the following is true: $E_{\lambda} \circ E_\lambda = id_\mathcal{V}$, $E_\lambda \circ E_{\lambda} = id_\mathcal{V}$. Thus it follows that $E_\lambda$ is an isomorphism of Banach spaces.

Let us denote

\begin{equation}
b(t, u, v) := a(t, u, v) + \lambda_0 < u, v >, \quad \text{where } \lambda_0 = c_0 + \frac{nc_1}{2} + 1
\end{equation}

Then $\forall u, v \in V \{t \mapsto b(t, u, v)\}$ is a measurable function and

\begin{align}
\exists c > 0 \quad & |b(t, u, v)| \leq c\|u\| \cdot \|v\|, \quad \forall u, v \in V, t \in [0, T] \\
\Re b(t, u, v) & \geq \|u\|^2, \quad \forall u \in V.
\end{align}
Hence the exists the unique \( B(t) \in \mathcal{L}(V, V') \):

\[
(B(t)u)(v) = b(t, u, v), \quad \forall u, v \in V
\]

Let us notice that \( B(t) = A(t) + \lambda_0 \cdot I \) where \( I: V \rightarrow V' \) is the imbedding from (6). Now we define the operators \( N, M \)

\[
N(v) := \{ t \mapsto B(t)v(t) \} \\
M(v) := \{ t \mapsto A(t)v(t) \}
\]

We shall prove

**Lemma 2.** \( M, N \in \mathcal{L}(V, V') \) i.e. \( M, N \) are well defined, bounded linear operators from \( L^2(O, T, V) \) to \( L^2(O, T, V') \).

**Proof of Lemma 2:** Because \( V' \) is isomorphic (in the category of Banach spaces) to Hilbert space \( V \), which is separable and reflexive thus \( V' \) is reflexive and separable, too. From Pettis' Theorem (see [5], p.131) we have that \( M(v) \) is strongly measurable (in Bochner sense) iff \( M(v) \) is weakly measurable (we have fixed \( v \in L^2(O, T, V) \)), i.e.

\[
\text{the function } \{ t \mapsto (Mv)(t)(u) \} \text{ is measurable. } \forall u \in V
\]

We shall prove (19). Let us observe that for any \( u \in V \) \((Mv)(t)(u) = (A(t)v(t))(u) = a(t, v(t), u)\). From the definition of Bochner measurable functions ([5], p.130) there exists a sequence of functions \( \{ v_n \}_{n \in \mathbb{N}} \), \( v_n : [O, T] \rightarrow V \), such that \( v_n \) is finitely valued (i.e.: \( \exists \{ B_j \}_{j=1}^s \) : \( B_j \) is Lebesgue measurable subset of \( [0, T] \), \( B_j \cap B_k = \emptyset \) if \( j \neq k \), \( v_n \) is constant on \( B_j \) \( j = 1, \ldots, s \) and \( v_n = 0 \) in the complement of \( \bigcup_j B_j \) and \( v_n(t) \rightarrow v(t) \) as \( n \rightarrow \infty \) a.e. on \( [0, T] \). Then, in view of (8), we get that \( a(t, v_n(t), u) \rightarrow a(t, v(t), u) \) a.e. on \( [0, T] \). Because the function

\[
[0, T] \ni t \mapsto a(t, v_n(t), u) = a \left( t, \sum_{j=1}^s z_j 1_{B_j}(t), u \right) = \sum_{j=1}^s 1_{B_j}(t)a(t, z_j, u)
\]

\((z_j = v_n|B_j)\) is measurable, thus \( \{ t \mapsto a(t, v(t), u) \} \) is measurable, too. Remaining conditions are in obvious way satisfied, so the proof is completed. ■

From the definitions of the spaces \( Y \) and \( V' \) it follows that the linear operator \( \Lambda \) (defined below in (20)) is continuous.

\[
\Lambda : Y \ni u \mapsto u' \in V'
\]
From the Theorems 1.1 and 4.1 of [3] (chapter 3) we get that the linear operator \( \Lambda + N \) is an isomorphism between \( X \) and \( \mathcal{V}' \).

**Remark:** The above statement is known and has been proved in [3].

Because \( \Lambda + N \) is a composition of the following maps:

(i) \[ X \ni u \mapsto \{ t \mapsto e^{-\lambda_0 t}u(t) \} \in X \quad ( = E_{-\lambda_0} ) \]

(ii) \[ X \ni w \mapsto (\Lambda + N)w \quad ( = \Lambda + N ) \]

(iii) \[ \mathcal{V}' \ni v \mapsto \{ t \mapsto e^{\lambda_0 t}v(t) \} \quad ( = E_{\lambda_0} ) \]

and each of them is an isomorphism, so

\[
(21) \quad \Lambda + M : X \longrightarrow \mathcal{V}'
\]

is an isomorphism.

The Trace Theorems (see [3] ch. 1, Th. 3.1, 3.2) give us that

\[
(22) \quad \gamma_0 : Y \ni u \mapsto u(0) \in H
\]

is a well defined bounded linear map, which is also onto.

Let us denote

\[
(23) \quad F : Y \ni u \mapsto ((\Lambda + M)u, u(0)) = (\Lambda + M, \gamma_0)(u) \in \mathcal{V}' \times H
\]

We shall prove

**Lemma 3.** \( F \) is an isomorphism

**Proof:** Because \( X = \{ u \in Y : \gamma_0 = 0 \} \) so in view of (21) \( F \) is an one-to-one map. Since \( F \) is a coutinous linear map, thus according to the Banach Open Mapping Theorem it is enough to show that \( F \) is onto. Let us take any \( f \in \mathcal{V}' \), \( u_0 \in H \). As \( \gamma_0 \) is onto there exists \( w \in Y : \gamma_0 w = u_0 \). Since \( \bar{f} := f - (\Lambda + M)w \in \mathcal{V}' \), from (21) we get that there exists \( \bar{u} \in X : (\Lambda + M)\bar{u} = \bar{f} \). If we put \( u := \bar{u} + w \) then \( \gamma_0 u = \gamma_0 \bar{u} + \gamma_0 w = u_0 \) and \( (\Lambda + M)u = \bar{f} + (\Lambda + M)w = f \).

In the previous part of this paper the functions \( a_i, b \) were fixed, operator \( M \) was defined in such a way that it was dependent on these functions. Let us denote \( a := (b, a_1, \ldots, a_n) \). We can define the functions \( M \) and \( F \) as follows:

\[
(24) \quad M : (L^\infty(\Omega_T))^{n+1} \ni a \mapsto Ma \in \mathcal{L}(Y, \mathcal{V}') \quad F : (L^\infty(\Omega_T))^{n+1} \ni a \mapsto Fa := ((\Lambda + M)a, \gamma_0a) \in \mathcal{L}(Y, \mathcal{V}' \times H)
\]
As we have shown in Lemma 3, for any \( a \in (L^\infty(\Omega_T))^{n+1} \) \( Fa \) is an isomorphism. That means, that for any \( f \in L^2(O,T,V') = L^2(O,T,H^{-1}(\Omega)) \) and for any \( u_0 \in H = L^2(\Omega) \) there exists the unique \( u \in Y \), such that \( u \) is a solution of the following problem:

\[
\begin{aligned}
\frac{du}{dt} + A(t)u(t) &= f(t) \\
u(0) &= u_0
\end{aligned}
\]  

(25)

Of course the map \((f, u_0) \mapsto u = F^{-1}(f, u_0)\) is analytic (because it is linear and continuous). We shall show, that \( u \) depends in a similar on the coefficient \( a \). To do this we define

\[
G: (L^\infty(\Omega_T))^{n+1} \times Y \ni (a, u) \mapsto F(a)u \in V' \times H
\]

(26)

**Lemma 4.** \( G \) is an analytic function and

\[
"d_{(a,u)}G = \frac{\partial G}{\partial u} (a,u) = F(a) \in \mathcal{L}(Y, V' \times H)
\]

(27)

**Proof:** Since \( G(a, u) = (M(a)u + \Lambda u, \gamma_0 u) \) it is enough to show that the map \((a, u) \mapsto M_1(a)u\) is analytic, where

\[
M_1(a)u := \sum_{i=1}^{n} a_i D_i u + bu = M(a)u + \Delta u
\]

(28)

(because \(-\Delta\) is a continuous linear map from \( Y \) to \( V' \))

The map in (28) is bilinear, so if we prove that it is continuous the proof will be finished. Hence we have to show that the maps

\[
L^\infty(\Omega_T) \times Y \ni (a, u) \mapsto a \cdot D_j u \in \mathcal{H} = L^2(O,T,H) \quad j = 1, \ldots, n
\]

\[
L^\infty(\Omega_T) \times Y \ni (b, u) \mapsto b \cdot u \in \mathcal{H}
\]

are continuous. But this follows from the continuity of the linear maps

\[
D_j: H^1_0(\Omega) \ni u \mapsto D_j u = \frac{\partial u}{\partial x_j} \in L(\Omega) = H
\]

\[
D_j: L^2(O,T,V) = L^2(O,T,H^1_0(\Omega)) \ni u \mapsto D_j u = \frac{\partial u}{\partial x_j} \in L^2(O,T,L^2(\Omega))
\]

In this way we have proved
**Theorem 1.** The map $G$ defined in (26) is analytic and its partial derivative with respect to $Y$ (at any point $(a, u)$) \(^{''}d_{(a, u)}G = F(a)\) is an isomorphism between $Y$ and $Y' \times H$.

We shall need the following version of the implicit function theorem

**Theorem 2.** Let us assume that $X, Y, Z$ are Banach spaces, $U$ is an open subset of $X \times Y, (a, b) \in U, f: U \to Z$ is of $\mathcal{C}^k$ ($k \in \mathbb{N} \cup \{\infty\}$) (or $f$ is analytic), $f(a, b) = c$

\[
^{''}d_{(a, b)}f = \frac{\partial f}{\partial y}(a, b) \in \mathcal{L}(Y, Z)
\]

is an isomorphism between $Y$ and $Z$. Then there exist:
- $K -$ a neighbourhood of $(a, b, c)$ in $U \times Z$
- $L -$ a neighbourhood of $(a, c)$ in $X \times Z$
- a function $g: L \to Z$ of class $\mathcal{C}^k$ (or analytic) such that

\[
(x, y, z) \in K, z = f(x, y) \Leftrightarrow (x, z) \in L, \quad y = g(x, z)
\]

Moreover

\[
\frac{\partial g}{\partial x}(a, b) = -\left(\frac{\partial f}{\partial y}(a, b)\right)^{-1} \circ \frac{\partial f}{\partial x}(a, b)
\]

\[
\frac{\partial g}{\partial z}(a, c) = \left(\frac{\partial f}{\partial y}(a, b)\right)^{-1}
\]

**Proof:** We shall use the classical version of this theorem (see [2], p.61).

Let us define:

\[
F: U \times Z \rightarrow Z \quad (x, y, z) \mapsto f(x, y) - z
\]

Then $F(a, b, c) = 0$ and $\frac{\partial F}{\partial y}(a, b, c) = \frac{\partial F}{\partial y}(a, b)$ is an isomorphism between $Y$ and $Z$. Hence there exist:
- $K -$ a neighbourhood of $(a, b, c)$ in $U \times Z$
- $L -$ a neighbourhood of $(a, c)$ in $X \times Z$
- a function $g: L \to Z$ of class $\mathcal{C}^k$ (or analytic) such that

\[
(x, y, z) \in K, z = f(x, y) \Leftrightarrow (x, z) \in L, y = g(x, z)
\]

and

\[
\frac{\partial g}{\partial z}(a, c) = -\left(\frac{\partial F}{\partial y}(a, b, c)\right)^{-1} \circ \frac{\partial F}{\partial z}(a, b, c) = \left(\frac{\partial f}{\partial y}(a, b)\right)^{-1}
\]

\[
\frac{\partial g}{\partial x}(a, c) = -\left(\frac{\partial F}{\partial y}(a, b, c)\right)^{-1} \circ \frac{\partial F}{\partial x}(a, b, c)
\]

\[
= -\left(\frac{\partial f}{\partial y}(a, b)\right)^{-1} \circ \frac{\partial f}{\partial x}(a, b)
\]

As a simple consequence we have the following
Corollary 1. Let \( f : X \times Y \rightarrow Z \) be a \( C^k \) (respectively analytic) map such that for any \( a \in X \), \( f(a, \cdot) \) is one to one from \( Y \) onto \( Z \), and \( \forall (x, y) \in X \times Y \) \( \frac{\partial f}{\partial y}(x, y) \) is an isomorphism between \( Y \) and \( Z \).

Then there exists a \( C^k \) (respectively analytic) function \( P : X \times Z \rightarrow Y \) such that

\[
P(x, z) = y \iff f(x, y) = z, \quad \forall (x, y, z) \in X \times Y \times Z
\]

Proof: If \( (a, c) \in X \times Z \) then there exists \( b \in Y \) : \( f(a, b) = c \). Let \( L, K, g \) be as in the conclusion of Theorem. We define

\[
P(x, z) := g(x, z) \quad \text{if} \ (x, z) \in L
\]

It is enough to show that \( P \) is well defined.

If there existed points \( (a_1, c_1), (a_2, c_2) \in X \times Z \) and their neighbourhoods \( L_1, L_2 \) such that for some point \( (x, z) \in L_1 \cap L_2 \) we had \( y_1 = g_1(x, z) \neq y_2 = g_2(x, z) \) then by (29) we would have \( z = f(x, y_1) \) and \( z = f(x, y_2) \). Thus obtained contradiction concludes the proof. \( \blacksquare \)

From Theorems 1 and 2 we get immediately the main result of this paper.

Theorem 3. The mapping

\[
P : L^2(O, T; H^{-1}(\Omega)) \times L^2(\Omega) \times (L^\infty(\Omega_T))^{n+1} \rightarrow Y
\]

that maps any triple \((f, u_0, a)\) into \( u \), the unique solution in \( Y \) of the problem (25), where \( Y \) is defined in (14), is analytic. It satisfies

\[
P(f, u_0, a) = u \iff G(a, u) = (f, u_0) \iff u \text{ is the unique solution to (25)}
\]

for any \( f, u_0, u \) in an appropriate space.

Corollary 2. Assume that \( E \) is an open subset of some Banach space and a function

\[
\Phi : E \ni \lambda \rightarrow (f(\lambda), u_0(\lambda), a(\lambda)) \in L^2(O, T; H^{-1}(\Omega)) \times L^2(\Omega) \times (L^\infty(\Omega_T))^{n+1}
\]

is a \( C^k \) (respectively analytic). For fixed \( \lambda \in E \) let \( u(\lambda) = u(t, x, \lambda) \in Y \) denotes the unique solution to the following, parametrized version of (25),

\[
u'(t, x, \lambda) - \Delta u(t, x, \lambda) + \sum_{j=1}^{n} a_j(\lambda) D_j u(t, x, \lambda) + b(\lambda) u(t, x, \lambda) = f(\lambda)
\]

\[
u(0, \cdot, \lambda) = u_0(\cdot, \lambda)
\]
Then the mapping $E \ni \lambda \rightarrow u(\lambda) \in Y$ is $C^k$ (respectively analytic).

Some extensions

1° We may consider a generalized version of equation (1) where the Laplacian $\Delta$ is replaced by a more general elliptic second order differential operator, i.e.:

$$
\frac{\partial u(t, x)}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(t, x) \frac{\partial u(t, x)}{\partial x_i} \right) \\
+ \sum_{i=1}^{n} a_i(t, x) \frac{\partial u(t, x)}{\partial x_i} + b(t, x) u(t, x) = f(t, x)
$$

(1')

together with the same initial and boundary conditions (2), (3).

We keep all the assumptions concerning the coefficients $a_i, b, f$ and moreover we assume that

$$
a_{ij} \in L^\infty(\Omega_T)
$$

(4')

$$
\exists c_0 > 0: \Re \sum_{i,j=1}^{n} a_{ij}(t, x) \xi_i \xi_j \geq |\xi|^2, \quad \forall \xi \in \mathbb{C}^n, \quad a.s. \text{ on } \Omega_T
$$

(4'')

(strong uniform ellipticity).

Then we define $a(t, \cdot, \cdot)$ by the formula

$$
a(t, u, v) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(t, x) D_j u(x) \overline{D_i v(x)} \, dx \\
+ \int_{\Omega} b(t, x) u(x) \overline{v(x)} \, dx + \sum_{i,j=1}^{n} \int_{\Omega} a_i(t, x) D_i u(x) \overline{v(x)} \, dx
$$

(7')

It is easy to see that (8), (9), (12) are satisfied. After defining $M$ as in (18) we can easily show that Lemma 3 is still valid.

Next we consider

$$
Q := \left\{ (a_{ij})_{i,j=1}^{n} \in (L^\infty(\Omega_T))^n : (4'') \text{ is satisfied} \right\}
$$

(37)

and observe that $Q$ is an open subset of $(L^\infty(\Omega_T))^n$. We define

$$
F: Q \times (L^\infty(\Omega_T))^n \ni ((a_{ij}), (a_i), b) \\
\rightarrow (\Lambda + M((a_{ij}), (a_i), b); \gamma_0 \in L(Y, \mathcal{V}'' \times H))
$$

(24')
If $G$ is defined as in (26) then Lemma 4 and Theorem 1 are still valid. Using Theorem 2 we get the following versions of Theorem 3 and Corollary 2.

**Theorem 4.** The mapping

$$
\tilde{P} : L^2(O,T; H^{-1}(\Omega)) \times L^2(\Omega) \times (L^\infty(\Omega_T))^{n+1} \to Y
$$

that maps any triple $(f, u_0, a)$ into $u$, the unique solution in $Y$ of the problem (1'), (2), (3), where $Y$ is defined in (14), is analytic. It satisfies

$$
P(f, u_0, a) = u \iff G(a, u) = (f, u_0) \iff u \text{ is the unique solution to (1'),(2),(3)}
$$

for any $f, a, u_0, u$ in the appropriate space.

**Corollary 3.** Assume that $E$ is an open subset of some Banach space and $\Phi : E \to W$, where $W = L^2(O,T; H^{-1}(\Omega)) \times L^2(\Omega) \times (L^\infty(\Omega_T))^{n+1}$ is a $C^k$(respectively analytic) mapping. Let for fixed $\lambda \in E, u(\lambda) \in Y$ denotes the unique solution to problem (1'), (2), (3), where $f, u_0, a_{ij}, a_i, b$ are replaced by $f(\lambda), u_0(\lambda), a_{ij}(\lambda), a_i(\lambda), b(\lambda)$ respectively and $\Phi(\lambda) = (f(\lambda), u_0(\lambda), (a_{ij}(\lambda)), (a_i(\lambda)), b(\lambda))$

then the map $E \ni \lambda \to u(\lambda) \in Y$ is $C^k$ (respectively analytic).

$2^o$ One can seek more regular solution to problem (1) – (3). In that case the functional spaces $Y, W, \ldots$ should be redefined.

For example let us show that Theorem 3 is still valid in the case of linearized Navier – Stokes Equations. We use the notation of [4]. Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$, with boundary $\partial \Omega$ of class $C^2$. We denote by $H$ and $V$ respectively the closure of $\{ \phi \in C_0^\infty(\Omega, \mathbb{R}^n) : div \phi = 0 \}$ in $(L^2(\Omega))^n$ and $(H^1_0(\Omega))^n$ respectively. Next we put $H^2 = V \cap (H^2(\Omega))^n$ and let $H^{1,2}(O,T)$ denotes the Hilbert space of all Bochner measurable function from $(O,T)$ into $H^2$ such that

$$
\int_0^T |u(t)|^2_{H_2} dt < \infty, \quad \int_0^T |u'(t)|^2_H dt < \infty
$$

where $u'(t)$ is the derivative of $u$ (as an $H$ valued function) in a distributional sense.

Let us denote by $\pi$ the orthogonal projection of $(L^2(\Omega))^n$ onto $H$.

We put

$$
\tilde{B}(u, v) := \frac{1}{2} \pi \left( \sum_{j=1}^n u^j D_j v + v^j D_j u \right), \quad u, v \in H^2
$$

$$
Au := -\pi \Delta u, \quad u \in H^2
$$
\( \tilde{B} \) is a bilinear continuous map from \( H^2 \times H^2 \) into \( H \), \( A \) is a bounded linear transformation from \( H^2 \) into \( H \), see [5].

For fixed \( \nu > 0, a \in \mathcal{H}^{1,2}(O,T), u_0 \in V \) and \( f \in L^2(O,T;H) \), we look for a function \( u \in \mathcal{H}^{1,2}(O,T) \) which is the solution to the following problem

\[
\begin{align*}
    u'(t) + \nu Au(t) + \tilde{B}(a(t), u(t)) &= f(t) \\
    u(0) &= u_0
\end{align*}
\]

(42)

It can be proved, see [1] for example, that a mapping \( (\Phi(a, \nu), \gamma_0) \) defined by

\[ \mathcal{H}^{1,2}(O,T) \ni u \rightarrow (u' + \nu Au + \tilde{B}(a,u), u(0)) \in L^2(O,T;H) \times V \]

is an isomorphism. Since the function

\[ G: (0, \infty) \times \mathcal{H}^{1,2}(0,T) \times \mathcal{H}^{1,2}(0,T) \ni (\nu, a, u) \rightarrow \Phi(a, \nu)u \in L^2(0,T;H) \]

is analytic and

\[ \frac{\partial G}{\partial u}(\nu, a, u) = \Phi(a, \nu) \]

we infer that the following is true

**Theorem 5.** The mapping

\[ P: L^2(0,T;H) \times V \times (0, \infty) \times \mathcal{H}^{1,2}(0,T) \rightarrow \mathcal{H}^{1,2}(0,T) \]

that maps any quadrupole \( (f, u_0, \nu, a) \) into \( u \), the unique solution in \( \mathcal{H}^{1,2}(0,T) \) of the problem (42), is analytic. It satisfies

\[ P(f, u_0, \nu, a) = u \Leftrightarrow G(\nu, a, u) = (f, u_0) \Leftrightarrow u \]

is the unique solution to (42) for any \( f \in L^2(0,T;H), (\nu, a) \in (0, \infty) \times \mathcal{H}^{1,2}(0,T), u_0 \in V, u \in \mathcal{H}^{1,2}(0,T) \).

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**References**


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