TRANSFINITE DIAMETER AND EXTREMAL POINTS
FOR A COMPACT SUBSET OF $\mathbb{C}^n$

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Abstract. In this paper we compute the transfinite diameter of a compact subset $K$ of $\mathbb{C}^n$ in terms of a new sequence of extremal points of $K$. This sequence is a generalization of Leja's sequence of extremal points of a compact subset $K$ of $\mathbb{C}$, which was defined in [2].

Introduction

Several notions of “capacity” have been introduced in the theory of functions of several complex variables. One of them is the transfinite diameter, which is defined for a compact subset $K$ of $\mathbb{C}^n$ as follows:

Let $e_1(z), e_2(z), \ldots$ be an enumeration of the monomials

$$z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n} \quad (\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n).$$

such that the degrees of the $e_j(z)$ are nondecreasing and the monomials $e_j(z)$ of a fixed degree are ordered lexicographically.

Let $h_s$ denote the number of the elements of the set \{ $j : \deg(e_j) = s$ \}. It is known that

$$h_s = \binom{s + n - 1}{n - 1}.$$

Let $m_s$ denote the number of the elements of the set \{ $j : \deg(e_j) \leq s$ \}. One can check that

$$m_s = \binom{s + n}{n}.$$

Let $x^{(k)} = \{x_1^{(k)}, x_2^{(k)}, \ldots, x_k^{(k)}\}$ (or shortly $\{x_1, \ldots, x_k\}$) denote a system of $k$ points $x_1, \ldots, x_k$ of $\mathbb{C}^n$. We define the generalized Vandermonde determinant $V(x^{(k)})$ of the point system $x^{(k)}$ by

$$V(x^{(k)}) := \det[e_i(x_j)]_{i,j=1,\ldots,k}.$$
Then $V(x^{(k)})$ is a polynomial in $x_1, \ldots, x_k$ of degree $l_s = \sum_{i=1}^{s} \deg(e_i)$. One may check that

$$\lambda_s := l_{m_s} = \sum_{j=0}^{s} j \cdot h_j = n \binom{s+n}{n+1}.$$  

A system $\xi^{(k)} = \{\xi_1^{(k)}, \ldots, \xi_k^{(k)}\}$ (or shortly $\{\xi_1, \ldots, \xi_k\}$) of $k$ points of a compact subset $K$ of $\mathbb{C}^n$ is called a system of extremal points of $K$ of order $k$, if

$$|V(\xi^{(k)})| = \sup \left\{ |V(x^{(k)})| : x^{(k)} \subset K \right\}.$$  

For such a system $\xi^{(k)}$ let us define:

$$V_k = V_k(K) := |V(\xi^{(k)})|$$

and

$$d_s(K) := V_s^{\frac{1}{s}}.$$  

For $n = 1$, Fekete proved in [1] that the limit

$$d(K) := \lim_{s \to \infty} d_s(K)$$

exists for any compact subset $K$ of $\mathbb{C}$. This limit is called the transfinite diameter of $K$.

In [5] Zaharjuta found a generalization of this fact, proving that the limit $d(K)$ exists for any compact subset $K$ of $\mathbb{C}^n$. His theorem gives an answer to an old problem posed by F. Leja (see [3]). This limit is also called the transfinite diameter of $K$.

For a compact subset $K$ of $\mathbb{C}^n$, let us consider two systems $\xi^{(i)}$ and $\xi^{(j)}$ ($i < j$) of extremal points of $K$ of order $i$ and $j$ respectively. In general it is not possible to choose them in such a way that $\xi^{(i)}$ be a subset of $\xi^{(j)}$. However, in [2] Leja defined another sequence of extremal points of a compact subset $K$ of $\mathbb{C}$. For this sequence it is true that $\xi^{(i)} \subset \xi^{(j)}$ for $i < j$. In the same paper Leja computed the transfinite diameter of a compact subset $K$ of $\mathbb{C}$ and the Green function of the unbounded component of $\mathbb{C} \setminus K$ with pole at infinity in terms of that sequence.

In this paper we define a similar sequence for a compact subset $K$ of $\mathbb{C}^n$ ($n \geq 1$) such that $\xi^{(i)} \subset \xi^{(j)}$ for $i < j$. We also compute the transfinite diameter of $K$ in terms of the sequence.
Main result

For a compact subset $K$ of $\mathbb{C}^n$, we define

$$M_i := \inf \left\{ \|p\|_K : p(z) = e_i(z) + \sum_{j=1}^{i-1} c_j e_j(z), \quad c_1, \ldots, c_{i-1} \in \mathbb{C} \right\},$$

$i \geq 1$

where the monomials $e_j(z)$ are defined as above ($\|p\|_K$ denotes the supremum norm of a function $p : K \to \mathbb{C}$ on $K$).

One can show that for every $i \geq 1$ there exists at least one $i$-th Tchebyseff polynomial $t_i(z)$ such that $\|t_i\|_K = M_i$.

We shall need the following

**Lemma 1.** (see [5], Lemma 4). Let $K$ be a compact subset of $\mathbb{C}^n$ such that $V_i > 0$ for $i \geq 1$. Then

$$M_i \leq \frac{V_i}{V_{i-1}} \leq i \cdot M_i, \quad i \geq 2.$$

**Proof:** Let $\{\xi_1, \ldots, \xi_{i-1}\}$ be a system of extremal points of $K$ of order $i - 1$. Then

$$V_{i-1} = |V(\xi_1, \ldots, \xi_{i-1})| > 0.$$

It is easy to see that the polynomial

$$P(z) := \frac{V(\xi_1, \ldots, \xi_{i-1}, z)}{V(\xi_1, \ldots, \xi_{i-1})}$$

has the following form:

$$P(z) = e_i(z) + \sum_{j=1}^{i-1} c_j e_j(z)$$

(in order to see it we can expand the determinant $V(\xi_1, \ldots, \xi_{i-1}, z)$ with respect to the last column). Hence

$$M_i \leq \|P\|_K = \frac{\sup \{|V(\xi_1, \ldots, \xi_{i-1}, z)| : z \in K\}}{V_{i-1}} \leq \frac{V_i}{V_{i-1}}.$$

Now let $\{\xi_1, \ldots, \xi_i\}$ be a system of extremal points of $K$ of order $i$. Then

$$V_i = |V(\xi_1, \ldots, \xi_i)| > 0.$$
Let \( t_i(z) \) denote the \( i \)-th Tchebysheff polynomial defined as above. Then

\[
V_i = \det \begin{bmatrix}
1 & 1 & \cdots & 1 \\
e_2(\xi_1) & e_2(\xi_2) & \cdots & e_2(\xi_i) \\
\vdots & \vdots & \ddots & \vdots \\
e_{i-1}(\xi_1) & e_{i-1}(\xi_2) & \cdots & e_{i-1}(\xi_i) \\
e_i(\xi_1) & e_i(\xi_2) & \cdots & e_i(\xi_i)
\end{bmatrix}
\]

\[
= \det \begin{bmatrix}
1 & 1 & \cdots & 1 \\
e_2(\xi_1) & e_2(\xi_2) & \cdots & e_2(\xi_i) \\
\vdots & \vdots & \ddots & \vdots \\
e_{i-1}(\xi_1) & e_{i-1}(\xi_2) & \cdots & e_{i-1}(\xi_i) \\
t_i(\xi_1) & t_i(\xi_2) & \cdots & t_i(\xi_i)
\end{bmatrix}
\]

We expand that determinant with respect to the last row and get

\[
V_i \leq \sum_{j=1}^{i} |t_i(\xi_j)| \cdot |V(\xi_1, \ldots, \xi_j-1, \xi_{j+1}, \ldots, \xi_i)| \leq i \cdot M_i \cdot V_{i-1}.
\]

The lemma is proved. \( \blacksquare \)

For a compact subset \( K \) of \( \mathbb{C}^n \), let us define a sequence of extremal points of \( K \) as follows.

Let \( a_1 \) be any point of \( K \) and let \( W_1 = 1 \). Suppose a system \( \{a_1, \ldots, a_{k-1}\} \) of points of the set \( K \) has already been constructed. Let us define the following polynomial:

\[
P_k(z) := V(a_1, \ldots, a_{k-1}, z) = \begin{bmatrix}
1 & \cdots & 1 \\
e_2(a_1) & \cdots & e_2(a_{k-1}) & e_2(z) \\
\vdots & \ddots & \vdots & \vdots \\
e_k(a_1) & \cdots & e_k(a_{k-1}) & e_k(z)
\end{bmatrix}
\]

Let \( a_k \) be any point of \( K \) such that

\[
|P_k(a_k)| = \sup \{|P_k(z)| : z \in K\}
\]

and let

\[
W_k := |P_k(a_k)|
\]

**Theorem 1.** For any compact subset \( K \) of \( \mathbb{C}^n \)

\[
\lim_{k \to \infty} (W_k)^{1/k} = d(K).
\]
Proof: It is obvious that $W_k \leq V_k$.

Now let us consider the following two cases:

a) The set $K$ is not unisolvent i.e. there exists a polynomial $p: \mathbb{C}^n \rightarrow \mathbb{C}$ such that $p = 0$ on $K$ but $p \neq 0$ in $\mathbb{C}^n$. Then it is easy to prove (using the Lagrange interpolation formula - see [4], Proposition 4.3) that there exists an $i_0$ such that for all $i \geq i_0$ $V_i = 0$. Then $d(K) = 0$ and $W_i = 0$ for all $i \geq i_0$, so the theorem is true.

b) The set $K$ is unisolvent. Then it is obvious that $W_k > 0$ for all $k$. Indeed, $W_1 = 1 > 0$, and if $W_{k-1} > 0$ then $P_k(z) = \sum_{i=1}^{k} c_i e_i(z)$ where $|c_k| = W_{k-1} > 0$, so $P_k \neq 0$ in $\mathbb{C}^n$. Therefore $P_k \neq 0$ on $K$ and so $W_k > 0$. Of course we also have $V_k > 0$ for all $k$. Hence for $i \geq 2$ the polynomial

$$P(z) = \frac{V(a_1, \ldots, a_{i-1}, z)}{V(a_1, \ldots, a_{i-1})}$$

is well defined and has the following form:

$$P(z) = e_i(z) + \sum_{j=1}^{i-1} c_j e_j(z).$$

Therefore

$$M_i \leq \sup \{|P(z)| : z \in K\} = \frac{W_i}{W_{i-1}}.$$

From Lemma 1 we derive

$$M_i \geq \frac{1}{i} \cdot \frac{V_i}{V_{i-1}},$$

and so

$$\frac{1}{i} \cdot \frac{V_i}{V_{i-1}} \leq \frac{W_i}{W_{i-1}}.$$

Hence

$$W_k = \frac{W_k}{W_{k-1}} \cdot \frac{W_{k-1}}{W_{k-2}} \cdots \frac{W_2}{W_1} \geq \frac{1}{k} \cdot \frac{V_k}{V_{k-1}} \cdot \frac{1}{k-1} \cdot \frac{V_{k-1}}{V_{k-2}} \cdots \frac{1}{2} \cdot \frac{V_2}{V_1} = \frac{V_k}{k!}.$$

We obtain

$$\left(\frac{V_k}{k!}\right)^{\frac{1}{k}} \leq (W_k)^{\frac{1}{k}} \leq (V_k)^{\frac{1}{k}}.$$
and now the result follows from Zaharjuta's theorem (see [5], Theorem 1):

\[ \lim_{k \to \infty} (V_k)^{\frac{1}{k}} = d(K) \]

and from the following obvious fact:

\[ \lim_{k \to \infty} (k!)^{\frac{1}{k}} = 1. \]

(In order to prove it we take \( s = s(k) \) such that \( m_{s-1} < k \leq m_s \). Then \( 1 \leq (k!)^{\frac{1}{k}} \leq (m_s^{m_s})^{\frac{1}{s-1}} \) and \( \lim_{s \to \infty} (m_s)^{\frac{m_s}{s-1}} = 1 \).)

REFERENCES

5. Zaharjuta, V. P., Transfinite diameter, Tchebycheff constants and a capacity of a compact set in \( C^n \), Mat. Sb. 96(138)(3) (1975), 374–389. (Russian)

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