NOTES ON CIRCULAR OPERATORS II

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Abstract. The present note concerns the commutation relation $e^{-itA}Te^{itA}$ $= e^{iat}T$, where $T \in L(H)$ and $A$ is a selfadjoint operator within the Hilbert space $H$. We prove several decompositions related to the interplay between $A$ and $T$.

1. We say that the operator $T \in L(H)$ is an $(A, \alpha)$ operator, if

(1.0) $e^{-itA}Te^{itA} = e^{iat}T$

for all real $t$, $A$ is selfadjoint, $\alpha$ is a real number and $\alpha \neq 0$. The one-parameter group $U(t) = e^{itA}$ is called the circulating group, and $T$ is called circular.

If $T$ is an $(A, \alpha)$ operator then:

(1.1) $e^{-itA}T^n e^{itA} = e^{i\alpha nt}T^n$, $t \in \mathbb{R}$,

(1.2) $e^{-itA}T^{-m} e^{itA} = e^{-i\alpha nt}T^{-m}$, $t \in \mathbb{R}$,

for $n, m = 1, 2, 3, 4, \ldots$. It follows that for all $t \in \mathbb{R}$ and $n, m = 1, 2, 3, \ldots$,

(1.3) $e^{-itA}T^{-m}T^n e^{itA} = e^{i\alpha(n-m)t}T^{-m}T^n$;

(1.4) $e^{-itA}T^n e^{itA} = e^{i\alpha(n-m)t}T^n T^{-m}$;

It follows from Prop. 1.0 of [9] that (1.3) and (1.4) are equivalent respectively to the following conditions:

(1.3') $T^{-m}T^n D(A) \subset D(A)$ and

$[T^{-m}T^n, A]f = \alpha(n - m)T^{-m}T^n f$ for $f \in D(A)$;

(1.4') $T^n T^{-m} D(A) \subset D(A)$ and

$[T^n T^{-m}, A]f = \alpha(n - m)T^n T^{-m} f$ for $f \in D(A)$. 

Notice that the operators \( T^n T^*n \) and \( T^*n T^n \) commute with \( A \), provided that \( T \) is an \((A, \alpha)\) operator.

Some examples of circular operators are now in order.

**Example 1.** Let \( H = \mathbb{C}^2 \) and take Pauli matrices

\[
\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Then \( e^{it \sigma_2} \sigma_+ e^{-it \sigma_2} = e^{it} \sigma_+ \) for all real \( t \).

**Example 2.** Let \( \{e_n\} \) \( n = 0, 1, 2, \ldots \) be the basis of the separable space \( H \). The quantum number operator \( N \) is defined by the equalities \( N e_n = n e_n \) \( (n = 0, 1, 2, \ldots) \) i.e. \( D(N) = \left\{ f : \sum_{n=0}^{\infty} n^2 |(f, e_n)|^2 < +\infty \right\} \) and \( N f = \sum_{n=0}^{\infty} n(f, e_n)e_n \). It follows that \( N \) is a selfadjoint operator. Since for \( n = 0, 1, 2, \ldots, e^{itN} e_n = e^{int} e_n \), we have \( e^{-itN} W e^{itN} e_n = e^{-itW} e_n \) for \( n = 0, 1, 2, \ldots \) where the bounded weighted shift \( W \) is defined by formulae \( W e_n = w_n e_{n+1} \) \( (n = 0, 1, 2, \ldots) \) with bounded sequence \( \{w_n\} \). It follows that \( W \) is a \((N, -1)\) operator.

**Example 3.** Suppose \( \mu \) is a positive measure on Borel subsets of \([0, 1]\), such that \( \mu(\{0\}) = \mu(\{1\}) = 0 \). We define \( \mu = \bigoplus_{n=0}^{\infty} \mu_n \) and write \( H = L^2(\mu; [0, +\infty)) \). Let the operator \( A \) be defined by the formula \((Af)(s) = sf(s)\) for \( f \in H \) such that \( \int_0^{\infty} |sf(s)|^2 \, d\mu_s < +\infty \). If \( V \) is defined as the unilateral shift i.e. \((Vf)(s) = 0\) for \( s \in [0, 1] \) and \((Vf)(s) = f(s - 1)\) for \( s \geq 1 \), then \( e^{-itA} Ve^{itA} = e^{-itV} \) - it follows that \( V \) is an \((A, -1)\) operator. Notice that \( \mu \) is "periodic" but can be equal to Lebesgue measure, as well as to a pretty singular measure living on Cantor set of positive Lebesque measure. It follows that the spectral type of the spectral measure of \( A \) is not determined by the commutation relation and the property that \( V \) is an isometry. If we take \( \bigoplus \mu_n = \mu \) and define \( A \) as the multiplication operator \(- (Af)(s) = sf(s)\), and \( V \) the bilateral shift i.e. \((Vf)(s) = f(s - 1), -\infty < s < +\infty\), then \( V \) is \((A, -1)\) operator and the spectral features of \( U(t) = e^{itA} \) are not unique. The moral is that there is no available analogon of J. V. Neumann's uniqueness theorem for Weyl's commutation relations for circular relations.
2. Suppose that the operator $T \in L(H)$ is an $(A, \alpha)$ operator. It follows then that for all real $t$

$$e^{-itA} (\Re T) e^{itA} = e^{-itA} \frac{1}{2} (T + T^*) e^{itA}$$

$$= \frac{1}{2} \left( e^{i\alpha t} T + e^{-i\alpha t} T^* \right)$$

$$= \frac{1}{2i} \left( i e^{i\alpha t} T + i e^{-i\alpha t} T^* \right).$$

When taking $t_0 = \frac{3}{2\alpha} \pi$ we infer that the following statement holds true:

(2.0) If $T$ is an $(A, \alpha)$ operator, then $\Re T$ and $\Im T$ are unitarily equivalent, namely

$$e^{-i \frac{3}{2\alpha} \pi A} \cdot \Re T \cdot e^{i \frac{3}{2\alpha} \pi A} = \Im T.$$

The above theorem generalizes a result of Ifantis [1], who proved if for weighted shifts by direct arguments, avoiding the commutation relation. His result follows from our Example 2 of the previous section and from (2.0) – (2.1). It follows now from (1.3) and (1.4) that the following property holds true;

(2.2) If $T \in L(H)$ is an $(A, \alpha)$ operator, then $\Re T^n T^{* -m}$ and $\Im T^n T^{* -m}$ are unitarily equivalent if $n \neq m$; $n, m = 1, 2, 3, \ldots$. In particular, $\Re T^n T^{* -m}$ and $\Im T^n T^{* -m}$ are equivalent, if $n \neq m$ and $T$ is a weighted shift,

Recall now that the operator $T \in L(H)$ is called semi-normal, if either $TT^* \leq T^* T$ or $T^* T \leq TT^*$.

We denote by $\sigma(Q)$ the spectrum of the operator $Q \in L(H)$.

Let us define the functions $p_X(z) = \Re z$, $p_Y(z) = \Im z$ for complex $z$. This is Putnam’s theorem – [4], [5], which reads as follows:

(P) If $T \in L(H)$ is a seminormal operator, then $\sigma(\Re T) = p_X(\sigma(T))$, $\sigma(\Im T) = p_Y(\Im T)$.

Recall now that the spectral radius $r(Q)$ of a semi-normal operator $Q$ equals to $\|Q\|$. Suppose now that $T$ is semi-normal and let $T$ be an $(A, \alpha)$ operator. Take $\lambda \in \sigma(T)$ such that $|\lambda| = \|T\|$. Since $T$ is circular, the circle $\{z: |z| = \|T\|\} \subset \sigma(T)$. It follows now from (P) and (2.0) that the following proposition holds true:

**Proposition 2.0.** Suppose that the semi-normal operator $T$ is an $(A, \alpha)$ operator. Then $\sigma(\Re T) = \sigma(\Im T) = [-r(T), r(T)]$.

The above proposition essentially generalizes the results of [1], [2], for special $T$, namely hyponormal weighted shifts of multiplicity one.
Suppose now that the seminormal operator $T$ is pure i.e. has no non-zero reducing normal part. This is the other theorem of Putnam – [4], [5], which says what follows:

$(P_0)$ If $T$ is a pure seminormal operator, then $\operatorname{Re} T$ and $\operatorname{Im} T$ have an absolutely continuous spectrum.

We derive therefore the important completion of Prop. 2.0 which reads as follows:

$(P_1)$ If $T$ is a pure seminormal $(A, \alpha)$ operator, then $\operatorname{Re} T$ and $\operatorname{Im} T$ have Lebesgue spectrum.

It follows from $(P_1)$ that for seminormal, pure $T$ of class $(A, \alpha)$ the parts $\operatorname{Re} T, \operatorname{Im} T$ have no point spectrum – this is an extension of results of [1], [2].

Some other spectral properties of circular operators are now in order. To begin with we define the parts of the spectrum of $T$ as follows:

\[
\begin{align*}
\sigma_p(T) &= \text{the point spectrum of } T; \\
\sigma_a(T) &= \text{approximate spectrum of } T; \\
\sigma_r(T) &= \text{residual spectrum of } T; \\
\sigma_c(T) &= \text{continuous spectrum of } T.
\end{align*}
\]

Suppose that $T$ is an $(A, \alpha)$ operator. Then for $t \in \mathbb{R}^1$ the operator $e^{it}T$ is unitarily equivalent with $T$. Since $\sigma_s(T)$ for $s = p, a, r, c$ is a unitary invariant, then the following proposition is true:

**PROPOSITION 2.1.** Suppose that $T \in L(H)$ is an $(A, \alpha)$ operator. Then for $s = p, a, r, c$ we have $e^{it}\sigma_s(T) = \sigma_s(T)$ for $t \in [0, 2\pi]$.

Simply, each part $\sigma_s(T)$, $s = p, a, r, c$ of the spectrum of $T$ is circled.

**CONCLUSION:** if $\dim H < +\infty$ then $\sigma(T) = \{0\}$.

This is the classical result of F. Riesz, which states what follows:

$(F)$ If $T \in L(H)$, $Tf = zf$ and $|z| = ||T||$, then $T^*f = \overline{zf}$; if $z_1 \neq z_2$, $|z_1| = |z_2| = ||T||$ and $Tf_k = z_kf_k$ ($k = 1, 2$), then $f_1 \perp f_2$.

**PROPOSITION 2.2.** Suppose that the nonzero operator $T \in L(H)$ is of class $(A, \alpha)$. Then there is no $z \in \sigma_p(T)$, such that $|z| = ||T||$ and consequently, no $u \in \sigma_r(T)$ such that $|u| = ||T||$.

**PROOF:** Suppose that $|z| = ||T||$ and $Tf = zf, f \in H$. Since

\[e^{-itA}Te^{itA}f = e^{it}Tf = e^{it}zf,\]
then \( T f(t) = e^{i\alpha t} z f(t) \), where \( f(t) = e^{it\Lambda} f \), for all real \( t \). We take \( \varepsilon > 0 \) sufficiently small and such that if \( 0 < |t| < \varepsilon \), then \( e^{i\alpha t} \neq 1 \), as well \( t_1 \neq t_2 \), \( 0 < |t_1| < \varepsilon \), \( 0 < |t_2| < \varepsilon \) and \( e^{i\alpha t_1} \neq e^{i\alpha t_2} \). It follows from (F) that \( f(t_1) \perp f(t_2) \). By taking \( t_1 \to 0 \), \( t_2 \to 0 \) we get that \( f(0) = f = 0 \).

To complete the proof, we notice merely that if \( u \in \sigma_r(T) |u| = \|T\| \), then \( \bar{u} \in \sigma_p(T^*) \). Since \( T^* \) is an \((A, -\alpha)\) operator, the assertion follows from the first part of our proposition.

**Remark 2.0.** If \( T \) is an \((A, \alpha)\) operator and some \( z \in \sigma_c(T) \) and \( |z| = \|T\| \), then \( \{u: |u| = \|T\|\} \subset \sigma_c(T) \).

The lack of the point spectrum on the circle \( \{z: |z| = \|T\|\} \) of the \((A, \alpha)\) operator \( T \in L(H) \) has some deep spectral reasons. Recall namely that by the celebrated Sz.-Nagy theorem – [7], [8], if \( T \) is a contraction i.e. if \( \|T\| \leq 1 \), then there is the unique semi–spectral measure \( F \) on Borel sets of the unit circle \( \Gamma \), such that

\[
(S) \quad T^n = \int_{\Gamma} z^n \, dF_z \quad \text{for} \quad n = 0, 1, 2, \ldots
\]

The last integral is the semi–spectral integral in the strong operator topology. Moreover, the part of point spectrum

\[
\sigma_p(T)_{\Gamma} = \{z: z \in \sigma_p(T), z \in \Gamma\}
\]

is characterized by the equality

\[
(2.3) \quad \sigma_p(T) = \{z: |z| = 1, F(\{z\}) \neq 0\} ;
\]

and for \( z \) such that \( |z| = 1 \) we have

\[
(2.4) \quad \operatorname{Ker}(zI - T) = \{f: F(\{z\})f = f\} .
\]

Supposing that the contraction \( T \) is an \((A, \alpha)\) operator, we derive from (S) that for \( f \in H \)

\[
(2.5) \quad \int_{\Gamma} z^n \, d (F_z e^{i\alpha t} f, e^{i\alpha t} f) = \int_{\Gamma} z^n \, d (F_z e^{i\alpha t} f, f) ,
\]

for \( n = 0, 1, 2, \ldots \) and the measure \( F_{\alpha, t}(\sigma) = F(e^{-i\alpha t} \sigma) \), for Borel sets \( \sigma \) on \( \Gamma \). Since the scalar measures in (2.5) are positive ones, then

\[
(2.6) \quad e^{-it\Lambda} F(\sigma) e^{it\Lambda} = F(e^{-i\alpha t} \sigma)
\]

for all \( t \) and Borel subsets \( \sigma \) of \( \Gamma \).
3. The present section deals with decomposition of circular operators. To begin with we recall the canonical decomposition of contractions — see [7], [8]. Let \( T \in L(H) \) be a contraction that is \( \|T\| \leq 1 \). We define the space

\[
H_n = \{ f : \|T^n f\| = \|f\| = \|T^* f\| \text{ for } n = 0, 1, 2, 3, \ldots \}
\]

The space \( H_u \) reduces \( T \) to the unitary operator \( T_u \). The subspace \( H_c = H \ominus H_u \) reduces \( T \) to the completely non-unitary operator \( T_c \), that is there is no non-zero subspace of \( H_c \) which reduces \( T \) to a unitary operator. It follows that \( T = T_u \oplus T_c \); this is the canonical decomposition of the contraction \( T \).

**Proposition 3.0.** Suppose that the contraction \( T \) is an \((A, \alpha)\) operator. Then \( H_u \) and \( H_c \) reduce \( A \) to \( A_u \) and \( A_c \) respectively and \( T_u \) is an \((A_u, \alpha)\) operator and \( T_c \) is an \((A_c, \alpha)\) operator.

**Proof:** Suppose that \( f \in H_u \). Since

\[
e^{-itA} T_u e^{itA} f = e^{i\alpha t} T^n f \quad \text{and} \quad e^{-itA} T^*_n e^{itA} f = e^{-i\alpha t} T^*_n f
\]

for \( n = 0, 1, 2, \ldots \), so \( \|T^*_n e^{itA} f\| = \|T^n f\| = \|f\| = \|T^*_n f\| = \|T^*_n e^{itA} f\| = \|e^{itA} f\| \), for each \( t \) and for any integer \( n \geq 0 \). It follows that \( H_u \) reduces \( e^{itA} \) for all \( t \). It follows that \( H_c \) reduces \( e^{itA} \) for all \( t \); Let \( A_u \) be the part of \( A \) in \( H_u \) and \( A_c \) the part of \( A \) in \( H_c \). It is plain now that \( T_u(c) \) is of \((A_u(c), \alpha)\) class, that is \( e^{-itA_u} T_u e^{itA_u} = e^{i\alpha t} T_u \) and \( e^{-itA_c} T_c e^{itA_c} = e^{i\alpha t} T_c \) for all \( t \).

Notice now that by the results of our first note, there is no unitary operator of class \((A, \alpha)\), if \( A \) is semi-bounded, that is \( A \) is bounded below or bounded above.

It follows now from Prop. 3.0 that the following theorem holds true:

**Theorem 3.0.** Suppose that the operator \( A = A^* \) is semi-bounded. Then every contraction of class \((A, \alpha)\) is completely non-unitary.

It is proved in [9], that a unitary \((A, -1)\) operator is a bilateral unitary shift. The proof uses the spectral theorem for \( A \); as the by – product of the proof one proves then easily our Th. 3.0. However, our proof here is spectral free. A natural problem arises, namely what about the isometric operators of \((A, \alpha)\) class, when \( A \) is semi-bounded. It follows from our above theorem that \( V \) must be a unilateral isometric shift. But by results of our first note, one infers that no isometric \((A, \alpha)\) operator exists, if \( H \neq 0 \) and \( A \) is bounded below and \( \alpha > 0 \). It follows then that the following theorem holds true:
Theorem 3.1. If the selfadjoint operator $A$ is bounded below, and $V$ is an $(A, \alpha)$ isometric operator, then $V$ is a unilateral isometric shift if $\alpha < 0$.

Now, if $A$ is bounded above and $e^{-itA}Ve^{itA} = e^{i\alpha t}V$ for all $t$, then $e^{-it(-A)}Ve^{it(-A)} = e^{-i\alpha t}V$ for all $t$. It follows then that $e^{-it(-A)}Ve^{it(-A)} = e^{i(-\alpha t)}V$ for all $t$. $B = (-A)$ is bounded below. By the above theorem we infer the following:

Corollary 3.0. If the isometric operator $V \in L(H)$ is an $(A, \alpha)$ operator and $A$ is bounded above and $\alpha > 0$, then $V$ is a unilateral isometric shift.

To complete our discussion concerning isometric circular operators, we assume that $V \in L(H)$ is a unilateral shift in $H$ and moreover $V$ is an $(A, \alpha)$ operator. Then $P_n = V^nV^* - V^{n+1}V^{*n+1}$ is the orthogonal projection on the subspace $H_n = V^n(H \ominus VH)$. Then the following conditions hold true:

(3.0) $H_n$ reduces $A$ for $n = 0, 1, 2, \ldots$ to the operator $A_n$;

(3.1) $A = \bigoplus_{n=0}^{\infty} A_n$;

(3.2) $e^{-itA_n+1}VP_ne^{itA_n}f = e^{-itA_n+1}Ve^{itA_n}P_nf = e^{i\alpha t}VP_nf$ (t arbitrary)

for all $f \in H$ and $n = 0, 1, 2, \ldots$;

(3.3) $A_n$ is unitarily equivalent to $(A_0 - n\alpha I)$ for $n = 0, 1, 2, \ldots$.

The above conditions complete the description of unilateral isometric shifts as $(A, \alpha)$ operators.

The next topic concerns the relationships between circularity and the canonical decompositions involving the "normality" properties of operators.

We assume that $T \in L(H)$ and define

$$H_n = \bigcap_{p,q \geq 0} \{f \in H : T^pT^*qf = T^*qT^pf\}.$$  

The subspace $H_n$ reduces the operator $T$ to a normal operator and the subspace $H_p \overset{df}{=} H \ominus H_n$ reduces $T$ to the pure operator $T_p$. The subspace $H_0 = \{f : Tf = 0 = T^*f\} \subset H_n$ reduces $T$ to the zero operator $T_0$. Then
\(T_n = \) the part of \(T\) in \(H_n \ominus H_0\) is an invertible normal operator. Summing up, we get that

\[(3.4)\]
\[T = T_0 \oplus T_n \oplus T_p ,\]

where \(T_0\) is a zero operator, \(T_n\) is an invertible normal operator and \(T_p\) is a pure operator.

Suppose now that \(T \in L(H)\) is an \((A, \alpha)\) operator. It follows that if \(f \in H_0\), then \(T e^{i\lambda t} f = 0\) and \(T e^{i\alpha t} f = 0\) for all real \(t\). Consequently \(H_0\) reduces \(A\). Let \(A_0\) be the part of \(A\) in \(H_0\).

If \(f \in H_n\), then by \((1.3)\) and \((1.4)\) we have by definition of \(H_n\) the equalities

\[T^{\ast q} T^p e^{i\lambda t} f = e^{-i\alpha(q-p)t} e^{i\lambda t} T^{\ast q} T^p f\]

\[T^p T^{\ast q} e^{i\lambda t} f = e^{i\alpha(p-q)t} e^{i\lambda t} T^p T^{\ast q} f\]

and \(T^p T^{\ast q} f = T^{\ast q} T^p f\) \((p,q = 0,1,2,\ldots)\). It follows that \(T^{\ast q} T^p e^{i\lambda t} f = T^p T^{\ast q} e^{i\lambda t} f\) for any \(p,q\) and for all \(t\). This means that \(H_n\) reduces \(e^{i\lambda t} A\), because \(H_0\) does. Indeed if \(T f = 0 = T^\ast f\) then since \(0 = e^{i\alpha t} T f = e^{-i\lambda A T e^{i\lambda t} A f}\)

and

\[0 = e^{-i\alpha t} T^\ast f = e^{-i\lambda A T^\ast e^{i\lambda t} A f},\]

then \(H_0\) reduces \(A\), say to \(A_0\). It follows now that \(H_p\) reduces \(A\). We denote by \(A_p\) the part of \(A\) in \(H_p\).

Summing up, we get that

\[(3.5)\]
\[A = A_n \oplus A_0 \oplus A_p\]

and

\[(3.6)\]
\[T = T_n \oplus T_0 \oplus T_p\]

where \(T_0\) is an \((A_0, \alpha)\) operator for \(s = n, 0, p\), and \(T_n\) is an invertible normal operator.

It is plain that nothing interesting happens with \(T_0\). In what follows we deal merely with \(T = T_n \oplus T_p\), where \(T_n\) is an invertible normal operator and \(T_p\) a pure one. This means that we assume that \(H = H_n \oplus H_p\).

Suppose that \(T \in L(H)\). Then \(T\) has the polar decomposition \(T = Q|T|\) where \(Q\) is a partial isometry and \(|T| = (T^\ast T)^{\frac{1}{2}}\). This decomposition is unique, if

\[(3.7)\]
\[\text{Ker } T = \text{Ker } |T| = \text{Ker } Q ;\]

We since now always assume that \((3.7)\) holds true. Suppose that \(T\) is an \((A, \alpha)\) operator. It follows then that \(T^\ast T\) commute with \(A\), and consequently \(|T|\) commutes with \(A\). It follows that \(\overline{R(|T|)}\) reduces \(e^{i\lambda A}\) and \(Q e^{i\lambda A}\) vanishes on \(\text{Ker } |T| = \text{Ker } Q\). It follows then that \(e^{-i\lambda A} Q e^{i\lambda A} |T| f = e^{i\alpha Q} |T| f\) for \(f \in H\), and consequently \(e^{-i\lambda A} Q e^{i\lambda A} g = e^{i\alpha Q} g\) for all \(g \in H\). Summing up we proved the following proposition:
Proposition 3.1. Let $T \in L(H)$ be of class $(A, \alpha)$ and let $T = Q|T|$ be the polar decomposition of $T$, where $\text{Ker } |T| = \text{Ker } Q = \text{Ker } T$. Then $Q$ is an $(A, \alpha)$ operator.

Remark. It is plain that the $\text{Ker } T = \text{Ker } |T| = \text{Ker } Q$ is almost needles for the knowledge of $T$ as a circular operator. It is then obvious that invertible circular operators are of some interest in the general theory. In this case $H = \overline{R(|T|)} = (\text{Ker } T)^\perp$. Suppose that the above $T$ equals to $T_n \oplus T_p$, where $T_n$ is an invertible normal operator and $T_p$ is pure. If $T$ is an $(A, \alpha)$ operator, then $T_n$ and $T_p$ are $(A, \alpha)$ operators. Let $T_n = Q_n |T_n|$ be the polar decomposition of $T_n$; $\text{Ker } Q_n = \text{Ker } T_n = \{0\} = \text{Ker } Q_n^* = \text{Ker } T_n^*$ because $T_n^{-1}$ and $T_n^{-1}$ exist. It follows that $Q_n$ is a unitary operator. Hence by Th. 2.1 of [6] and Prop. 3.1 we arrive to the following theorem:

Theorem 3.2. Let $T \in L(H)$ be a non zero $(A, \alpha)$ operator and $T = T_n \oplus T_p$, where $T_n$ is normal and invertible, and $T_p$ is pure. Then, if $A$ is semi-bounded, then $T_n = 0$ i.e. $T = T_p$ i.e. $T$ must be pure.

4. The Prop. 3.1 has some others interesting consequences, namely the following theorem:

Theorem 3.3. Suppose that $T \in L(H)$ is a non zero $(A, \alpha)$ operator and $\alpha > 0$ and $A$ is bounded below. Then if $T^{-1}$ exists, then $T = |T^*| S_d^*$, where $S_d$ is a unilateral isometric shift of multiplicity equal to $d = \text{dim } \text{Ker } T$, with the wandering subspace equal to $\text{dim } \text{Ker } T$, $\text{Ker } T$ being the wandering subspace.

Proof: $T$ is not invertible, because otherwise the partial isometry $Q$ of polar decomposition of $T$ would be unitary, which contradicts results of [6]. If $T = Q|T|$ is the polar decomposition of $T$ then $T^* = Q^*|T^*|$ is the polar decomposition of $T^*$. Since $\text{Ker } T^* = \{0\}$ by assumption, the operator $Q^*$ is an isometry. It follows from results in [6] and Prop. 3.1, that $Q^*$ is completely non-unitary – it follows that $Q^*$ is a unilateral shift $S_d$ with wandering subspace equal to $(I - Q^*Q)H$. But $Q^*Q$ is the orthogonal projection on $\overline{R(|T|)}$; it follows that $(I - Q^*Q)$ is the orthogonal projection on $\text{Ker } T$. It follows that $T^* = Q^*|T^*| = S_d^*|T^*|$. Hence $T = |T^*| S_d^*$ which, since $(I - Q^*Q)$ is the orthogonal projection on $\text{Ker } T$, proves our claim.

For other analogous of Th. 3.1 see [3]. Lastly, we derive from Prop. 3.1 and results of [6] the following proposition:

Proposition 3.2. Suppose that the operator $A$ is selfadjoint and semi-bounded. Then, if $T \in L(H)$ is an $(A, \alpha)$ operator, then zero is not in the resolvent set of $T$, that is $0 \in \sigma(T)$. 


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