On the Differentiability of Regular Solutions of a Linear Functional Equation

In the present note we shall deal with the linear functional equation

\[ \varphi[f(x)] = g(x)\varphi(x) + h(x), \]

where the given functions \( f, g, \) and \( h \) are real-valued functions of a real variable, defined in an interval \( (0, a), 0 < a \leq +\infty, \) zero being the unique fixed point of the function \( f(x) \) in this interval. In our previous paper [2] we have found some conditions of the existence of the only one-parameter family of so called regular solutions \( \varphi(x) \) of equation (1) in the case where there exists a continuous solution depending on an arbitrary function. A solution of equation (1) is called regular if it is continuous in the interval \( [0, a) \) and has the derivative at the point zero.

In the present paper we give some conditions that every regular solution of equation (1) be differentiable in the whole interval \( [0, a) \). As in [2], we shall consider the indeterminate case \( g(0) = f'(0) = 1 \).

We shall make the following assumptions:

(i) The function \( f(x) \) is of class \( C^1 \) in \( [0, a) \), strictly increasing in \( [0, a) \), \( f(0) = 0, \) \( 0 < f(x) \) in \( (0, a) \), \( f'(0) = 1 \).

(ii) The function \( g(x) \) is continuous in \( [0, a) \) and belongs to class \( C^1 \) in \( (0, a) \), \( g(x) > 0 \) in \( (0, a) \), \( g(0) = 1 \).

(iii) The function \( h(x) \) is of class \( C^1 \) in \( [0, a) \), \( h(0) = 0 \).

Let us remark that the condition \( h(0) = 0 \) is the necessary condition of the existence of continuous solutions of equation (1) in the interval \( [0, a) \) (it is enough to put \( x = 0 \) in equation (1)).

First of all, let us note a lemma on solutions of equation (1) that are of class \( C^1 \) in the open interval \( (0, a) \) (cf. [1] and [3]).
Lemma 1. Suppose hypotheses (i)—(iii) to be fulfilled. Then equation \( (1) \) possesses in the interval \((0, a)\) a \( C^1 \)-solution depending on an arbitrary function, i.e., every function \( \varphi(x) \), defined and of class \( C^1 \) in an interval \([f(x_0), x_0]\), \( x_0 \in (0, a) \), and fulfilling the conditions
\[
\varphi[f(x_0)] = g(x_0)\varphi(x_0) + h(x_0),
\]
\[
\varphi'[f(x_0)]f'(x_0) = g(x_0)\varphi'(x_0) + g'(x_0)\varphi(x_0) + h'(x_0)
\]
can be uniquely extended to a solution \( \varphi(x) \) of equation \( (1) \) in the whole interval \((0, a)\). The function \( \varphi(x) \) is of class \( C^1 \) in \((0, a)\).

We denote by \( f^n(x) \) the \( n \)-th iterate of the function \( f(x) \) and we put
\[
(2) \quad p_n(x) = \frac{d}{dx} [f^n(x)] = \prod_{i=0}^{n-1} f'[f^i(x)].
\]

We shall need estimates of the sequences \( f^n(x) \) and \( p_n(x) \).

Lemma 2. Let hypothesis (i) be fulfilled and assume that there are positive numbers \( b_1, k, \mu \) such that in the interval \([0, a)\) the function \( f(x) \) can be written in the form
\[
(3) \quad f'(x) = 1 - b_1 x^k + R(x), \quad R(x) = O(x^{k+\mu}), \quad x \to 0+0.
\]

Then

1° for an arbitrary \( x_1 \in (0, a) \) and for every positive number \( \varepsilon \), one can find a positive integer \( N = N(\varepsilon, x_1) \) such that for \( n > N \) we have
\[
(4) \quad f^n(x_1) > K n^{-1/k}, \quad K \overset{\text{def}}{=} [(a_1 + \varepsilon) k]^{-1/k}, \quad a_1 \overset{\text{def}}{=} b_1/(k+1).
\]

2° there exists an \( x \in (0, a) \) such that for an arbitrary \( x_1 \in (0, x) \) holds the inequality
\[
(5) \quad p_n(x) < M n^{-a}
\]
where \( M = M(x_1) \) is a positive constant and \( a \) is a number fulfilling the condition
\[
(6) \quad a > \alpha \overset{\text{def}}{=} \max(1, 1/k).
\]

The inequality \( (5) \) holds for \( n \) sufficiently large and for every \( x \in [f(x_1), x_1] \).

Proof. Because of assumptions \( (3) \) we can write the function \( f(x) \) in the form
\[
f(x) = x - a_1 x^{k+1} + R_1(x), \quad R_1(x) = O(x^{k+1+\mu}), \quad x \to 0+0,
\]
where \( a_1 \) is defined in \( (4) \). One can easily verify that the function \( f(x) \) thus fulfills the assumptions of Theorem 3.1. of Thron’s paper [4], from which the statement 1° of the Lemma follows.

To prove 2° take an \( \bar{x} \in (0, a) \) such that in the interval \([0, \bar{x}]\) we have the inequality
\[
f'(x) < 1 - rx^k,
\]
where

\[ \frac{b_1}{\kappa (k+1)} < r < b_1, \]

and \( \kappa \) is defined by (6). Since we have assumed (3), we can really find such an \( \bar{x} \). Next, let \( x_1 \) be an arbitrarily fixed point from the interval \((0, \bar{x})\). We have

\[ f' [f^i(x)] < 1 - r [f^i(x)]^k < 1 - r [f^{i+1}(x_1)]^k \]

for \( x \in [f(x_1), x_1] \) and \( i = 0, 1, \ldots \). We take an \( \varepsilon \) such that

\[ 0 < \varepsilon < \frac{r}{k \kappa} - a_1 \]

(the right-hand member of the above inequality is a positive number, cf. (7)) and we make use of \( 1^\circ \) of our Lemma. The inequality (4) together with (8) yields for \( i > N \)

\[ f' [f^i(x)] < 1 - \frac{a}{i+1} \]

where

\[ a = r [a_1 + \varepsilon] k^{-1} > \kappa \geq 1 \]

(cf. (9)), and therefore we have

\[ \left(1 - \frac{a}{i+1}\right) < \left(1 - \frac{1}{i+1}\right)^a. \]

Now we can write the estimate of \( p_n(x) \) for \( n > N \) and \( x \in [f(x_1), x_1] \),

\[ p_n(x) = p_N(x) \prod_{i=N}^{n-1} f' [f^i(x)] < p_N(x) \prod_{i=N}^{n-1} \left(1 - \frac{1}{i+1}\right)^a < M n^{-a}, \]

where \( M = M(x_1) \overset{\text{def}}{=} M^*(x_1) N^{-a} \) and \( M^*(x_1) = \sup_{[f(x_1), x_1]} p(x) \). This, together with (10), completes the proof of the Lemma.

Now we are going to prove a theorem on the differentiability of regular solutions of the homogeneous linear equation

\[ \varphi [f(x)] = g(x) \varphi (x), \]

which corresponds to Theorem 1 of [2].

Put

\[ g_n(x) = \prod_{i=0}^{n-1} g[f^i(x)], \]

and

\[ \gamma_n(x) = \frac{f^n(x)}{g_n(x)}. \]
Theorem 1. Suppose, besides (i) and (ii), that the function $f'(x)$ fulfils (3) of Lemma 2 and that there exists a positive number $v$ such that we can write the function $g'(x)$ as

$$g'(x) = -a_1 kx^{k-1} + R_2(x), \quad R_2(x) = O(x^{k-1 + v}), \quad x \to 0+0.$$  \hspace{1cm} (14)

Then equation (11) has the only one-parameter family of solutions that are differentiable in the interval $[0, a)$. They are given by the formula

$$\varphi_\varepsilon(x) = \varepsilon \lim_{n \to \infty} \gamma_n(x)$$  \hspace{1cm} (15)

(cf. (13) and (12)). The functions (15) are of class $C^1$ in $(0, a)$.

Proof. From Theorem 1, [2], we infer that under our assumptions (stronger than those in Theorem 1, [2]) the sequence $\gamma_n(x)$ uniformly converges in every interval $[0, \delta], \delta > 0$, that functions $\gamma_n(x)$ are the unique regular solutions of equation (11). We have to prove that the function $\varphi_\varepsilon(x) = \lim_{n \to \infty} \gamma_n(x)$ has the continuous derivative in an interval $[f(x_1), x_1]$. Thus, let us form the sequence

$$\frac{d\gamma_n(x)}{dx} = \gamma_n(x)[P_n(x) - Q_n(x)]$$  \hspace{1cm} (16)

where

$$P_n(x) = p_n(x)[f^n(x)], \quad Q_n(x) = \sum_{i=0}^{n-1} \frac{g'[f^i(x)]}{g[f^i(x)]} p_i(x),$$  \hspace{1cm} (cf. (2), (12), (13)).

Let $x_1$ be an arbitrary point of the interval $(0, \bar{x})$, $\bar{x}$ being fixed in $2^\circ$ of Lemma 2, and let us take an $\varepsilon > 0$ and the $N$ named in $1^\circ$ of this Lemma such that inequalities (14) and (5) hold for $n > N$ and $x \in [f(x_1), x_1]$.

For the sequence $P_n(x)$ we can write the estimate

$$0 < P_n(x) < \frac{p_n(x)}{f^{n+1}(x_1)} < K_0 (n+1)^{-a-1/k}, \quad n > N,$$  \hspace{1cm} (17)

for $x \in [f(x_1), x_1]$, where $K_0$ is a positive constant depending on $x_1$.

For the sequence $Q_n(x)$ we shall majorize the absolute value of a term of the sum, in the interval $[f(x_1), x_1]$, of course. Because of assumption (14), we can find an $x_1 \in (0, \bar{x})$ such that in the interval $(0, x_1]$ we have

$$|g'(x)| < K_1 x^{k-1},$$

where a positive number $K_1$ depends on the $x_1$. We assume that we have taken the same $x_1$ as in (17) above. According to hypothesis (ii) the function $g(x)$ has in $[0, x_1]$ a positive lower bound, say $K_2$. Thus, for every positive integer $i$, we have

$$\frac{|g'[f^i(x)]|}{g[f^i(x)]} < \frac{K_1}{K_2} [f^i(x)]^{k-1}, \quad x \in [f(x_1), x_1].$$  \hspace{1cm} (18)
Two cases are possible.  

1) $0 < k < 1$. Then we have 

\[ |f(x)|^{k-1} < [f(x)]^{k-1} \text{ for } x \in [x_1, x_2]. \]

Now, by the use of (4) and (5), we obtain from (18), for $i > N$, the inequality 

\[ \frac{|g[f(x)]|}{g[f(x)]} p_i(x) < \frac{K_1}{K_2} K^{k-1} (i+1)^{-\left(k-1\right)/k} M_i^{-\alpha} > K^* i^{-\beta_0}, \]

where $\beta_0 \defeq 1 + a - 1/k > 1$ on account of (6), since in this case we have $\alpha = 1/k$.

2) $k \geq 1$. From assumption (14) we can deduce that the function $g(x)$ is bounded in $[0, x_1]$, say $|g(x)| < K_2$ for $x \in [0, x_1]$. Thus we obtain by (5) for $i > N$ and $x \in [x_1, x_2]$ the inequality 

\[ \frac{|g[f(x)]|}{g[f(x)]} p_i(x) < \frac{K_3}{K_1} M_i^{-\alpha}, \]

where $\alpha > 1$ (cf. (6)).

Inequality (17) implies the uniform convergence of the sequence $P_n(x)$, in the interval $[f(x_1), x_2]$ (to zero). Inequalities (19) or (20) imply the uniform convergence of the sequence $Q_n(x)$ in the same interval. This, together with the uniform convergence of the sequence $\gamma_n(x)$ yields the uniform convergence of the sequence (16) in this interval. The function $\varphi_1(x)$ restricted to the interval $[f(x_1), x_2]$ is of class $C^1$. It may be easily verified that this function fulfils conditions of Lemma 1. Consequently, there exists the unique solution of equation (11) in $(0, 1)$, belonging to class $C$ in $(0, 1)$, which is the extension of the function $\varphi_1(x)$ restricted to the interval $[f(x_1), x_2]$. In other words: the function $\varphi_1(x)$ is a $C^1$-solution of equation (11) in $(0, 1)$. But $\varphi_1(x)$, as a regular solution of (11), has also a derivative at the point zero. Thus all the functions (15) fulfil the desired conditions and the proof is completed.

Further, let us put 

\[ \psi(x) = \sum_{n=0}^{\infty} h[f^n(x)]/G_{n+1}(x) \]

(cf. (12)). We are going to prove a theorem on the differentiability of regular solutions of equation (1) (cf. [2], Theorem 2).

**Theorem 2.** If the assumptions of Theorem 1 are fulfilled and there exists a positive number $\lambda$ such that 

\[ h'(x) = O(x^{k+1}), \ x \to 0+, \]

and the function $h(x)$ fulfills hypothesis (iii), then equation (1) has the unique one-parameter family of solutions, differentiable in the interval $(0, a)$, given by the formula 

\[ \varphi(x) = \varphi_1(x) + \psi(x), \]
where the functions \( q_c(x) \) (\( c \) is an arbitrary real number) are defined by (15) and
the function \( \nu(x) \) is defined by (21). The functions (23) are of class \( C^1 \) in \((0, a)\).

Proof. Since the assumptions of Theorem 2, [2], are weaker than our assumptions, the theorem is applicable. Thus we know that the functions (23) are the unique regular solutions of equation (1) in \([0, a)\). Simultaneously, it follows from Theorem 1 (just proved) that the functions \( q_c(x) \) are differentiable
in \([0, a)\) and belong to class \( C^1 \) in \((0, a)\). Consequently, it is enough to prove that the function (21) has the same properties.

Let us denote by \( a_n(x) \) the \( n \)-th term of the series (21). The sequence \( a_n(x) \)
may be written in the form

\[
a_n(x) = \frac{h[f^n(x)]}{f^{n+1}(x)} \gamma_{n+1}(x).
\]

We differentiate the sequence \( a_n(x) \) and form the series

\[
(24) \quad \sum_{n=0}^{\infty} a'_n(x) = \sum_{n=0}^{\infty} \left\{ \gamma_{n+1}(x) \left[ \frac{h'[f^n(x)]}{f^{n+1}(x)} p_n(x) - \frac{h[f^n(x)]}{(f^{n+1}(x))^2} p_{n+1}(x) \right] + \gamma'_{n+1}(x) \frac{h[f^n(x)]}{f^{n+1}(x)} \right\}.
\]

We shall prove that the series (24) uniformly converges in an interval \([f(x_1), x_1]\),
\( x_1 \in (0, a) \). For this purpose, first of all, let us take an \( x \in (0, \bar{x}] \), where \( \bar{x} \) is defined
in Lemma 2, and let the inequality

\[
(25) \quad |h'(x)| < Lx^{k+\lambda}
\]

hold for \( x \in [0, x_1] \), where \( L \) is a positive constant. Because of assumption (22),
such an \( L \) and an \( x_1 \) actually exist. Next, let us take an \( \varepsilon, 0 < \varepsilon < a_1 \), and let
us choose an \( N = N(\varepsilon, x_1) \) such that inequality (4) holds for \( n > N \), and inequality (5) holds for \( n > N \) and \( x \in [f(x_1), x_1] \). At last, we make the convention
that for \( n > N \) also the inequality

\[
(26) \quad f^n(x_1) < L_n^{-1/k}, \quad L_n \overset{\text{def}}{=} [(a_1 - \varepsilon)^{k-1/k}]
\]

holds. Inequality (26) follows also from Theorem 3.1 of [4], similarly as (4).
We shall estimate separately the components of the \( n \)-th term of series (24),
except of \( \gamma_{n+1}(x) \) and its derivative.

We obtain by (25), (4) and (26)

\[
\frac{|h'[f^n(x)]|}{f^{n+1}(x)} < \frac{L[f^n(x_1)]^{k+\lambda}}{f^{n+2}(x_1)} < \frac{LL_n n^{-1/k}}{K(n+2)^{-1/k}}, \quad n > N, \quad x \in [f(x_1), x_1].
\]

Hence, by (5), there exists a positive constant \( A_1 \), such that

\[
(27) \quad \frac{|h'[f^n(x)]|}{f^{n+1}(x)} p_n(x) < A_1 n^{-\beta_1}, \quad \beta_1 \overset{\text{def}}{=} 1 + \alpha + (\lambda - 1)/k > 1
\]
for \( n > N \) and \( x \epsilon [f(x_1), x_1] (\beta_1 > 1\) owing to inequality (6) for \( a \). Further, assume that in the interval \([0, x_1]\) holds the inequality

\[
|h(x)| < La^{k+1+1},
\]

which is a consequence of (25) and (iii). Then there is an \( A_3 > 0 \) such that for \( n > N \) and \( x \epsilon [f(x_1), x_1] \) we have

\[
\frac{|h[f^n(x)]|}{|f^{n+1}(x)|} p_{n+1}(x) < \frac{LLM n^{-1-(i+1/k)(n+1)-a}}{K^2 (n+2)^{-2/k}} < A_3 n^{-\beta_1}
\]

according to (4) and (26). Finally, we have also the inequality

\[
\frac{|h[f^n(x)]|}{f^{n+1}(x)} < A_3 n^{-\beta_2}, \quad \beta_2 \overset{\text{def}}{=} 1 + \lambda/k > 1,
\]

(with an \( A_3 > 0 \)) valid for \( x \epsilon [f(x_1), x_1] \) and \( n > N \).

On the other hand, from the proof of Theorem 1 above, it follows that the sequences \( \gamma_{n+1}(x) \) and \( \gamma'_n(x) \) converge uniformly in the interval \([f(x_1), x_1]\) and thus they are bounded in this interval. Hence, according to (27)–(29), the inequality

\[
|a'_n(x)| < An^{-\beta}
\]

follows, with an \( A > 0 \) and \( \beta > 1 \), valid for \( n \) sufficiently large and for every \( x \epsilon [f(x_1), x_1] \). Thus the series (24) has the majorant \( \Sigma A \alpha n^{-\beta} \) and, consequently, the function \( \psi(x) \) defined by (21) belongs to class \( C^1 \) in the interval \([f(x_1), x_1]\). By the use of Lemma 1, arguing similarly as in the proof of Theorem 1, we obtain the desired conclusion: the function \( \psi(x) \) is differentiable in \([0, a)\) and of class \( C^1 \) in \((0, a)\). This completes the proof.

Finally, let us note the following theorem that is an immediate consequence of Theorem 3, [2] and of Lemma 1.

**Theorem 3.** Suppose that the function \( f(x) \) fulfills the assumptions of Lemma 2, and the function \( g(x) \) can be written in the form

\[
g(x) = 1 - a_2 x^k + R_5(x), \quad R_5(x) = 0(x^{k+r}), \quad x \rightarrow 0 + 0,
\]

where \( a_2 > 0, a_2/a_1 = b_1/(k+1) \), and hypothesis (ii) is fulfilled.

Then in the case \( a_1 < a_2 \) equation (11) possesses in the interval \([0, a)\) a differentiable solution depending on an arbitrary function, while in the case \( a_1 > a_2 \) the function \( \varphi(x) \equiv 0 \) is the unique solution of equation (11) that is differentiable in \([0, a)\).

**References**


