On Singular Solutions of a Matrix Multiplicative Functional Equation

1. The well known Cauchy's equation for matrices

\[ F(AB) = F(A)F(B), \]

where \( A, B \) denote the \( n \times n \) matrices, \( F \) is an \( m \times m \) matrix-function, has in the case where \( A, B \) and \( F \) are non-singular matrices and \( m < n \) the following general solutions (cf. [3])

\[ F(A) = \Phi(J)CAC^{-1}, \quad J = \det A, \]

\[ F(A) = \Phi(J)C(A^{T})^{-1}C^{-1}, \]

\[ F(A) = G(J), \]

\( C \) being a non-singular constant matrix, \( \Phi(J) \) and \( G(J) \) scalar and arbitrary matrix-functions of one scalar argument, respectively, satisfying the equation

\[ G(xy) = G(x)G(y). \]

In the present note we shall give the general solutions of equation (1) in the case where \( m < n \) but where the matrices \( A, B \) or the matrix \( F \) may be singular. The elements of these matrices are from an arbitrary field \( K \). Let us denote by \( GL(n) \) the multiplicative semigroup of all square \( n \times n \) matrices over \( K \), whereas \( GL(n) \) denotes as usual the full group of such non-singular matrices. There exist four possibilities for the multiplicative function \( F(A) \):

I \( F: GL(n) \rightarrow GL(m) \)

II \( F: GL(n) \rightarrow \overline{GL}(m) \)

III \( F: \overline{GL}(n) \rightarrow GL(m) \)

IV \( F: \overline{GL}(n) \rightarrow \overline{GL}(m) \).
We shall deal with the singular cases II-IV. The solution in the case II for \( n = m = 2 \) has been given by Kucharzewski and Kuczma [2].

We recall that if a function \( F(A) \) is a solution of (1) and \( C \) is a non-singular matrix then \( CF(A)C^{-1} \) is also a solution of (1) and thus it is sufficient to determine the solution \( F(A) \) with accuracy to the above similarity relation, i.e. to the choice of a base for \( F \) as linear transformation of the vector space \( K^m \).

First we shall state several lemmas which are valid for arbitrary \( m \) and \( n \).

Lemma 1. For any family of commuting idempotent matrices there exist a base in which all these matrices have a diagonal form (see [1], p. 15).

Let us denote by \( \{d_1, \ldots, d_n\} \) a diagonal matrix with the elements \( d_1, \ldots, d_n \) on the main diagonal.

Lemma 2. The following matrices
\[
A_{1\ldots r} = \{1_{1\ldots r}, 0_{0\ldots 0}\} \quad 0 < r < n - 1
\]

and
\[
\begin{align*}
R(q) &= \{q, 1_{1\ldots 1}\} \\
S(q) &= \left\{ \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix}, 1_{1\ldots 1} \right\} \\
V_i &= \left\{ 1_{1\ldots 1}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1_{1\ldots 1} \right\},
\end{align*}
\]

where the submatrix \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) stands in lines \( i, i+1 \), generate the full semigroup \( GL(n) \), i.e. any matrix \( A \in GL(n) \) is a product of a finite number of such matrices.

In fact, matrices (5) generate all elementary non-singular matrices and thus the full linear group \( GL(n) \subset \overline{GL}(n) \). If a matrix \( B \) is singular of a rang \( r \) then there exist non-singular matrices \( P, Q \) such that
\[
B = PA_{1\ldots r}Q.
\]

\( P, Q \) are generated by matrices (5) and thus \( B \) is generated by matrices (4) and (5).

Definition 1. A solution \( F(A) \) of equation (1) will be called decomposable if there exists a base in which it has the form
\[
F(A) = \begin{bmatrix} F_1(A) & 0 \\ 0 & F_2(A) \end{bmatrix},
\]
i.e. if it is the direct sum of solutions \( F_1 \) and \( F_2 \).

Definition 2. A solution \( F \) of equation (1) will be called regular if \( F(A) = 0 \) for any singular \( A \) and \( F(A) \in GL(m) \) for any non-singular \( A \).

Obviously any regular solution is a trivial extension of a solution of equation (1) in the case I, by putting \( F(A) = 0 \) for any singular matrix \( A \).

Lemma 3. If \( F(A) \) is a non-decomposable solution then
\[
F(0) = 0 \quad \text{and} \quad F(E_n) = E_m,
\]
where $E_n, E_m$ are the unit matrices of order $n$ and $m$ respectively, and 0 is the null matrix.

Proof. The matrices $F(0)$ and $F(E_n)$ are idempotent, since 0 and $E_n$ are idempotent. If $F(0) = J 
eq 0$, then, by lemma 1, in a suitable base $J$ has the form

$$J = \{1...1, 0...0\}$$

(it is obvious that only 0's and 1's may stand on the main diagonal). Any matrix $A$ commutes with the null matrix and thus $F(A)$ commutes with $J$, what implies that the solution $F(A)$ must be decomposable. The same reasoning leads to this conclusion in the case where $F(E_n) = J = 0$.

Remark. In view of (1) any constant solution different from zero is an idempotent matrix and thus it must have the form

$$F(A) = C^{-1}JC,$$

where $C$ is a constant non-singular matrix and $J$ has one of forms (8).

Lemma 4. The restriction of any non-decomposable solution $F: \overline{GL}(n) \to \overline{GL}(m)$ to the group $\overline{GL}(n)$ is a solution of the type I and thus if $n < m$ it has one of forms (2).

The above statement follows easily from the equation $F(E_n) = E_m$ which implies that $F(\Delta A) = E_m$ for any $A \in \overline{GL}(n)$ and thus $F(A)$ is also a non-singular matrix, and moreover

$$F(\Delta A^{-1}) = F^{-1}(A).$$

Relation (10) allows us to state the following.

Lemma 5. If the matrices $A, B \in \overline{GL}(n)$ are similar then the matrices $F(A)$ and $F(B)$ are also similar, provided $F$ is a non-decomposable solution.

2. In this section we shall determine all solutions of equation (1) in the most general case IV. For the sake of brevity we shall write $\tilde{A}$ instead of $F(A)$. Thus, for $A \in \overline{GL}(n)$ we will have $\tilde{A} \in \overline{GL}(m)$.

By $A_{i_1...i_r}$ we denote the diagonal matrix with r 1's standing on the diagonal on places $i_1 < ... < i_r$. All such matrices form a family of commuting idempotents and, in view of (1), all corresponding $\tilde{A}_{i_1...i_r}$ form such a family. By Lemma 1, we can choose a base in which they have the diagonal forms

$$\{\varepsilon_1, ... , \varepsilon_m\},$$

where, by idempotence, $\varepsilon_i = 0$, 1.

In the sequel we shall assume that the considered solution $F$ is non-decomposable and thus, in particular, that relations (7) hold. We assume also $1 < m < n$.

Since for the matrices $A_i$ ($i = 1, ..., n$), $A_i A_i = 0$ for $i \neq j$, we have also

$$\tilde{A}_i \tilde{A}_j = 0 \quad \text{for} \quad i \neq j$$

(11)
(by (7), \( \tilde{A} = 0 \)). All \( A_i \) and, by Lemma 5, also all \( \tilde{A}_i \) are similar and consequently they have the same rank, say \( s_1 \).

It follows from (11) that the matrices \( \tilde{A}_i, \tilde{A}_j (i \neq j) \) have the 1's on different places. But for \( m < n \) this may happen only if \( s_1 = 0 \) and for \( m = n \) only if \( s_1 = 0, 1 \).

The case \( m = n \). Suppose first that \( s_1 = 0 \), i.e.

\[
\tilde{A}_i = 0 \quad \text{for} \quad i = 1, \ldots, n.
\]

Set \( s_2 = \text{rang} \tilde{A}_{ij} \) \((i, j = 1, \ldots, n)\). Since, for \( (ij) \neq (kl) \), \( A_{ij} A_{kl} = A_{jk} \delta_{i, l} \), using (12) we get

\[
\tilde{A}_{ij} \tilde{A}_{kl} = 0.
\]

The number of all matrices \( \tilde{A}_{ij} \) equals \( \binom{n}{2} > n \), for \( n > 2 \). For \( n = 2 \) we need only the statement about \( \tilde{A}_1 \), because in this case it is \( A_{12} = E \) and, by (7), \( A_{12} = E_{12} \). In the same way as in the precedent case we conclude from (13) that \( s_2 = 0 \). Analogously we can prove that \( s_3 = \text{rang} \tilde{A}_{ijk} \) vanishes if \( n > 3 \) and so on. Generally,

\[
\tilde{A}_{i_1 \ldots i_r} = 0 \quad \text{for} \quad 0 < r < n-1.
\]

Using formula (6), by (14) we get

\[
\tilde{B} = \tilde{F} \tilde{A}_{i_1 \ldots i_r} \tilde{Q} = 0,
\]

so that we have \( F(B) = 0 \) for every singular matrix \( B \in GL(n) \), and thus the solution \( F \) is regular (see definition 2).

Suppose now \( s_1 = 1 \). In this case we can obtain, in view of the equality \( m = n \), that

\[
\tilde{A}_i = A_i \quad \text{for} \quad i = 1, \ldots, n.
\]

From the corresponding relations for the matrices \( A_{ij} \) and \( A_i \), by addition of the sign ",,,~" and using formula (15), we get

\[
\tilde{A}_{ij} \tilde{A}_k = \begin{cases} A_k & \text{if} \quad k = i \text{ or } j \\ 0 & \text{if} \quad k \neq i, j \end{cases}
\]

Hence we conclude easily that \( \tilde{A}_{ij} = A_{ij} \) \((i, j = 1, \ldots, n)\). And generally, from the formulas

\[
\tilde{A}_{i_1 \ldots i_r} A_k = \begin{cases} A_k & \text{if} \quad k \text{ is one of indices } i_1, \ldots, i_r \\ 0 & \text{otherwise} \end{cases}
\]

we conclude that

\[
\tilde{A}_{i_1 \ldots i_r} = A_{i_1 \ldots i_r}.
\]
Now we shall determine $F$ for a non-singular $A$. From the obvious relations

$$B(e)A_i = \begin{cases} B_i(e) & \text{if } i = j \\ A_j & \text{if } i \neq j \end{cases}$$

where $B_i(e) = \{1, \ldots, \sigma \ldots, 1\}$, in view of (1) and (15) we get

$$B_i(e)A_j = \begin{cases} B_i(e) & \text{if } i = j \\ A_j & \text{if } i \neq j \end{cases}$$

and hence it follows immediately that

$$(17) \quad \tilde{B}_i(e) = B_i(\psi(e))_{i, i} \quad i = 1, \ldots, n.$$  

We write the same function $\psi(e)$ for all indices $i$ since all matrices $B_i(e)$ and thus, by lemma 5, all images $\tilde{B}_i(e)$ are similar for a fixed value of $e$.

Furthermore, using the corresponding relations for $S(e)$ and $A_i$, by (15) we obtain

$$(18) \quad \tilde{S}(e)A_i = A_i, \quad A_i\tilde{S}(e) = A_i, \quad A_i\tilde{S}(e)A_i = A_i \quad \text{for } i = 3, \ldots, n,$$

where $S(e)$ is defined by (5). Hence we conclude that $\tilde{S}(e)$ has the form

$$(19) \quad \tilde{S}(e) = \begin{bmatrix} 1 & a(e) \\ 0 & 1 \end{bmatrix}, 1, \ldots, 1$$

and, from the equality $\tilde{S}(e + \sigma) = \tilde{S}(e)\tilde{S}(\sigma)$, it follows that $a(e)$ is an additive function, i.e. $a(e + \sigma) = a(e) + a(\sigma)$.

On the other hand, from the relation $S(e) = B_i(e)S(1)\tilde{B}_i^{-1}(e)$ written for images, in view of (17) and (19) we get

$$(20) \quad a(e) = \psi(e)a, \quad a = a(1)$$

and thus $\psi(e)$ is an additive function. But evidently $\psi(e)$ is a multiplicative function as it follows from equality $\tilde{B}_i(e\sigma) = \tilde{B}_i(e)\tilde{B}_i(\sigma)$. It is well known, however, that a multiplicative and additive scalar function of one real variable is the identity, i.e. $\psi(e) = e$.

Consequently, by (20), we have $a(e) = ea$ and substituting it into (19) we get

$$\tilde{S}(e) = \begin{bmatrix} 1 & ea \\ 0 & 1 \end{bmatrix}, 1, \ldots, 1$$

Taking instead of $F$ the solution $C^{-1}FC$ where $C = \{a, 1, \ldots, 1\}$ for the new solution $F$ we have

$$(21) \quad \tilde{S}(e) = S(e)$$
and, as \( \varphi(q) = q \),

\[
\tilde{B}_i(q) = B_i(q), \quad i = 1, \ldots, n.
\]

Furthermore, from the obvious relations

\[
\tilde{V}_i A_i \tilde{V}_i = A_{i+1} \quad \text{and} \quad A_k \tilde{V}_i A_k = A_k \quad \text{if} \quad k \neq i, i+1
\]

we obtain

\[
\tilde{V}_i = \begin{bmatrix} 1, \ldots, 1, \frac{0}{c_i}, 1, \ldots, 1 \end{bmatrix}, \quad i = 1, \ldots, n-1.
\]

In particular, from the equality \( A_2 \tilde{V}_1 S(1) A_2 = A_1 \) and from (21) we get \( c_1 = 1 \). Taking in turn instead of \( F \) the solution \( D^{-1}FD \), where

\[
D = \{ 1, 1, c_2, c_2 c_3, \ldots, c_2 \ldots c_{n-1} \}
\]

for the transformed solution \( F \) we have

\[
\tilde{V}_i = V_i \quad \text{for} \quad i = 1, \ldots, n,
\]

and formulas (21) and (22).

Thus we have proved that there exists a base in which the considered solution \( F \) satisfies formulas (16), (21), (22) and (23), what means that \( F \) is the identity on the set of generators (4) and (5). Consequently \( F \) is the identity on the whole semigroup \( GL(n) \), i.e. \( F(A) = A \) for \( A \in GL(n) \). Considering \( F \) in an arbitrary base we get generally

\[
F(A) = C^{-1}AC.
\]

Accordingly, we have

**Lemma 6.** In the case \( m = n \), any non-decomposable solution of equation (1) of type IV either has the general form (24) or is regular.

**Remark.** In order to determine \( F \) on the group \( GL(n) \) (i.e. for a non-singular \( A \)) we could use, by Lemma 4, the known solutions (2), but in this method the computations seem longer.

The case \( m < n \). As we have already proved, in this case \( s_1 = \text{rang} \tilde{A}_i = 0 \), i.e. \( \tilde{A}_i = 0 \). Using further the same procedure as in the case \( m = n \) (valid also if \( n > m \)) we obtain formula (14) from which we conclude, as in the precedent case, that the considered solution is regular. Therefore we have

**Lemma 7.** In the case \( 1 < m < n \) any non-decomposable solution of equation (1) of type IV is regular.

**Remark.** We have obtained the above results using only formulas (7); therefore the statements of Lemmas 6 and 7 are valid also if we assume only (7).

If \( F \) is a decomposable solution of (1) then it is a direct sum of a finite number of non-decomposable solutions of orders \( < n \) which, by Lemma 7, are
regular if their orders are \( > 1 \), and may be constant, 0 or 1, if they are one-dimensional. Evidently any one-dimensional and non-constant solution is regular. Thus, taking into account that the direct sum of any regular solutions is also a regular one, we can write any non-constant solution different from (24) in the form

\[
F(A) = C^{-1} \begin{bmatrix}
F_1(A) \\
1 \\
1 \\
0 \\
0
\end{bmatrix} C
\]

(25)

may be \( F_1(A) = F(A) \), where \( F_1(A) \) is regular.

\( F_1(A) \) restricted to the group \( GL(n) \) is a solution of equation (1) of type I, and thus, in the considered case \( m < n \), it has one of forms (2). Accordingly we have

\[
\begin{align*}
F_1(A) &= 0 & \text{if } A \text{ is a singular matrix} \\
F_1(A) &= \text{is defined by (2) for the non-singular arguments.}
\end{align*}
\]

(26)

In particular, if \( F_1 \) is of order \( m_1 < n \) then it may take only the form (2c), i.e. \( F_1(A) \) depends only on the determinant \( J = \det A \). Assuming by definition \( G(0) = 0 \), we have in this case

\[
F_1(A) = G(J), \quad J = \det A, \quad A \in GL(n).
\]

(27)

Consequently we can formulate our main theorem as follows

**Theorem.** Any matrix function \( F: GL(n) \to GL(m) \) (\( m < n \)) satisfying the functional equation (1) has one of forms (9), (24) (only if \( m = n \)) or (25), where the function \( F_1 \) is given by (26) or (27).

In the case II, i.e. \( F: GL(n) \to GL(m) \), we get only the trivial solution

\[
F(A) = E.
\]

In fact, \( F(0) \) is a non-singular idempotent and thus it equals \( E \), from which we get

\[
F(A) = F(0)F(A) = F(0A) = F(0) = E.
\]

In the case III, i.e. \( F: GL(n) \to GL(m) \), the matrix \( F(E) \) must be singular, since in the other case any matrix \( F(A) \) would be non-singular and we would have the case I. If \( F(E) = J \) and \( r = \text{rang } J \), then in a suitable base \( J \) has the form (8) and from the relations

\[
F(A)J = JF(A) = F(A)
\]
we get

\[ F(A) = \begin{bmatrix} F_1(A) & 0 \\ 0 & 0 \end{bmatrix}, \]

where \( F_1(A) \) is an \( r \times r \) matrix and \( F_1(E) = E_r \), what implies that all values \( F_1(A) \) are non-singular. Thus \( F_1 \) is a solution of type I.

Of course, the above two statements are valid for arbitrary \( m, n \).

REFERENCES