A Generalization of the Second Theorem of O. Hanner

In his paper [3] Hanner gives the proofs of two theorems, called respectively first and second theorem of Hanner, concerning the notion of the $\text{ANR} (\mathbb{R})$-spaces.

First theorem of Hanner. Every open subset of an $\text{ANR} (\mathbb{R})$-space is an $\text{ANR} (\mathbb{R})$-space.

Second theorem of Hanner. If a metrizable space $X$ is the countable union of open sets $G_i$ ($i = 1, 2, ...$) which are $\text{ANR}(\mathbb{R})$-spaces, then $X$ is an $\text{ANR}(\mathbb{R})$-space.

The definitions of the $\text{ANR}(\mathbb{R})$-space and the proofs of the theorems of Hanner are to be found in [1] (Chapter IV, p. 85-99), where the following problem is raised: Is it true that a metrizable space $X$ in which every point has a neighbourhood being an $\text{ANR}(\mathbb{R})$ is necessarily an $\text{ANR}(\mathbb{R})$?

This paper contains a positive answer to this question.

First we shall prove the following.

Lemma 1. If a metrizable space $X$ is the union of his open pairwise disjoint subsets being the $\text{ANR} (\mathbb{R})$-spaces, then $X$ is an $\text{ANR} (\mathbb{R})$-space.

Proof. We may assume that $X$ is a closed subset of a metric space $Y$. We have

$$X = \bigcup_{\nu \in M} G_{\nu}, \quad \nu \neq \mu \Rightarrow G_{\nu} \cap G_{\mu} = \emptyset, \quad G_{\nu} \in \text{ANR} (\mathbb{R}),$$

where $G_{\nu}$ are open in $X$ and $M$ denotes an arbitrary set of indexes. Suppose first that there exists a family $\{U_{\nu}\}_{\nu \in M}$ such that $U_{\nu} \supset G_{\nu}, U_{\nu}$ are open in $Y$, and $U_{\nu}$ are pairwise disjoint. We have

$$G_{\nu} = X - \bigcup_{\mu \neq \nu} G_{\mu}.$$
hence $G_\ast$ is a closed subset of $Y$. $G_\ast$ being an $ANR(\mathcal{M})$-space, we may find a retraction $r_\ast: V_\ast \to G_\ast$, where $V_\ast$ is an open subset of $Y$. Taking

$$W_\ast = U_\ast \cap V_\ast, \quad i_\ast = r_\ast|_{W_\ast}, \quad \text{and} \quad i_\ast = \bigcup_{\ast \in M} i_\ast,$$

we have a retraction

$$i_\ast: \bigcup_{\ast \in M} W_\ast \to X.$$

If the family $\{G_\ast\}_{\ast \in M}$ satisfies the condition: $(\ast)$ there exists a number $d > 0$ such that $\nu \neq \mu \Rightarrow d(G_\ast, G_\mu) > d$, then $X$ is an $ANR(\mathcal{M})$ since we may take as $\{U_\ast\}_{\ast \in M}$ the sets

$$U_\ast = \bigcup_{x \in G_\ast} K(x, d/3),$$

$K(x, r)$ being the open ball with center $x$ and radius $r$.

In the general case we take

$$G_{\ast i} = \{x \in G_\ast: d(x, X - G_\ast) > 1/i\}, \quad \nu \in M \quad (i = 1, 2, \ldots).$$

Since $G_{\ast i}$ is open in $G_\ast$, $G_{\ast i}$ is an $ANR(\mathcal{M})$ by the first theorem of Hanner. We can easily verify that the family $J_\ast = \{G_{\ast i}\}_{\ast \in M}$ satisfies the condition $(\ast)$. Hence

$$\bigcup_{\ast \in M} G_{\ast i}$$

is an $ANR(\mathcal{M})$, and

$$X = \bigcup_{i=1}^{\infty} \bigcup_{\ast \in M} G_{\ast i}$$

is an $ANR(\mathcal{M})$ by the second theorem of Hanner. The proof of lemma 1 is thus completed.

Now let us pass to the general case. $X$ being metric space, it is paracompact; therefore we may assume that

$$X = \bigcup_{\ast \in M} G_\ast, \quad G_\ast \text{ open in } X, \quad G_\ast \in ANR(\mathcal{M})$$

and that $\{G_\ast\}_{\ast \in M}$ is locally finite.

We define by induction a sequence $G^k = \{G^k_\ast\}_{\ast \in M} \ (k = 0, 1, \ldots)$ of open and locally finite coverings of the space $X$. For $k = 0$,

$$G^0 = \{G_\ast\}_{\ast \in M} = \{G^0_\ast\}_{\ast \in M}.$$

The covering $G^k = \{G^k_\ast\}_{\ast \in M}$ being defined, we define a covering $G^{k+1} = \{G^{k+1}_\ast\}_{\ast \in M}$ as an open and locally finite covering of the space $X$ satisfying the condition

$$G^{k+1}_\ast \subset G^k_\ast.$$

That such a covering exists we may deduce from the very well known theorem on paracompact spaces ([2], p. 209).
For every covering $\mathcal{G}^k$ and for every point $x \in X$ we define a positive integer $a_k(x)$ by the conditions: $a_k(x) = n$ if and only if

(i) There exists a sequence of $n$ sets $G^k_{r_1}, \ldots, G^k_{r_n}$ such that for every neighbourhood $V_x$ of $x$ we have $G^k_{r_i} \cap V_x \neq \emptyset$ ($i = 1, \ldots, n$).

(ii) There exists a neighbourhood $V^k_x$ such that the sets $V^k_x \cap G^k_{r_i}$ are non-empty only for $v = r_i$ ($i = 1, \ldots, n$).

It is evident that $a_k(x)$ is well defined for every point $x \in X$ and for every covering $\{G^k_{r_i}\}_{r_i \in M}$.

Let

$$K^k_n = \{x \in X : a_k(x) = n\}$$

and

$$A^k_{r_1, \ldots, r_n} = \bigcap_{i=1}^n G^k_{r_i} \cap \bigcup_{x \in K^k_n} V^k_x$$

where $(r_1, \ldots, r_n) \in M^n$ and $V^k_x$ is a neighbourhood of $x$ satisfying the condition (ii) of the definition of $a_k(x)$. We have:

a) $A^k_{r_1, \ldots, r_n}$ is open for every $(r_1, \ldots, r_n) \in M^n$ and $k = 1, 2, \ldots$

b) $A^k_{r_1, \ldots, r_n} \subseteq \bigcap_{i=1}^n G^k_{r_i}$, so that $A^k_{r_1, \ldots, r_n}$ is an $ANR(\mathcal{G})$.

c) The sets $A^k_{r_1, \ldots, r_n}$ are pairwise disjoint.

To prove c) let us suppose that $\mu_i \neq r_i$ for every $i = 1, \ldots, n$ and let there exist a point $y \in X$ such that

$$y \in A^k_{r_1, \ldots, r_n} \cap A^k_{\mu_1, \ldots, \mu_n}.$$ 

Since $y \in A^k_{r_1, \ldots, r_n}$, there exists a point $x \in K^k_n$ such that

$$y \in V^k_x \cap \bigcap_{i=1}^n G^k_{r_i}. $$

By the definition of $V^k_x$, the neighbourhood $V^k_x$ has empty intersections with all the sets $G^k_v$, $v \neq r_i$. But $y \in V^k_x \cap G^k_{\mu_1}$ and $\mu_1 \neq r_i$. This is impossible, and this implies that the sets $A^k_{r_1, \ldots, r_n}$ are pairwise disjoint.

Let

$$A^k_n = \bigcup_{(r_1, \ldots, r_n) \in M^n} A^k_{r_1, \ldots, r_n}. $$

Lemma 1 implies that $A^k_n$ is an $ANR(\mathcal{G})$. By the second theorem of Hammer we have that

$$\mathcal{B}^k = \bigcup_{n=1}^\infty A^k_n$$

is an $ANR(\mathcal{G})$.

Now we shall prove the following

Lemma 2. For every $x \in X$,

$$x \notin \mathcal{B}^k \Rightarrow a_{k+1}(x) < a_k(x). $$

Proof. Suppose that \( a_k(x) = n \), and let \( G^1_{r_1}, \ldots, G^n_{r_n} \) be the sets satisfying the definition of \( a_k(x) \). By the definition of \( a_k(x) \) we have

\[ x \in \overline{G^k_{r_i}} \ (i = 1, \ldots, n). \]

Since \( x \notin \mathcal{B}^k \), we have

\[ x \notin \bigcap_{i=1}^{n} G_{r_i}^k. \]

Hence there exists an index \( r_j \), \( 1 < j < n \), such that

\[ x \in \overline{G_{r_j}^k} \quad \text{and} \quad x \notin G_{r_j}^k. \]

Let us take now the sets \( G_{r_i}^{k+1} \) \( (i = 1, \ldots, n) \). It is evident that \( a_{k+1}(x) < a_k(x) \) since the point \( x \) may lie at most in the sets \( G_{r_i}^{k+1} \). But \( x \notin \overline{G_{r_j}^{k+1}} \) since if \( x \in \overline{G_{r_j}^{k+1}} \), we have by the definition of the covering \( \{G_{r_i}^{k+1}\}_{r \in M} \) that

\[ x \in \overline{G_{r_j}^{k+1}} \subset G_{r_j}^k \]

and this is impossible. Hence \( a_{k+1}(x) < a_k(x) \) if \( x \notin \mathcal{B}^k \).

Now to complete the proof of our result it is sufficient to show that

\[ X = \bigcup_{k=1}^{\infty} \mathcal{B}^k. \]

The inclusion

\[ \bigcup_{k=1}^{\infty} \mathcal{B}^k \subset X \]

being evident, we may assume that there exists a point \( x \in X \) such that

\[ x \notin \bigcup_{k=1}^{\infty} \mathcal{B}^k. \]

So, for every \( k = 1, 2, \ldots, x \notin \mathcal{B}^k \), and by lemma 2 \( \{a_k(x)\} \) is a strongly decreasing sequence of positive integers. But this is impossible and the proof is completed.

REFERENCES