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GENERIC PROPERTIES OF ITERATED FUNCTION SYSTEMS WITH PLACE DEPENDENT PROBABILITIES

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Abstract. It is shown that most (in the sense of Baire category theory) Iterated Function Systems with place dependent probabilities are asymptotically stable and nonexpansive.

Introduction. Let $S_i \colon X \to X$, $i = 1, ..., N$ be a sequence of transformations and let $p_i : X \to [0, 1]$ be a probabilistic vector. The action of the iterated function system (IFS) with state dependent probabilities, $(S_1, ..., S_N; p_1, ..., p_N)$, can be roughly described as follows. We choose an initial random element $x_0 \in X$ and then we randomly select an integer from the set $\{1, ..., N\}$ in such a way that the probability of choosing k is $p_k(x_0)$, $k = 1, ..., N$. When a number k_0 is drawn, we define $x_1 = S_{k_0}(x_0)$. Next we select k_1, according to the probabilistic vector $(p_1(x_1), ..., p_N(x_1))$ and we define $x_2 = S_{k_1}(x_1)$ and so on. Denoting by μ_n the distribution of x_n, i.e.

$$\mu_n(A) = \text{prob}\{x_n \in A\} \quad \text{for every non-negative integer } n$$

we are interested in a convergence of measures μ_n to a measure μ_* independent of the initial measure μ_0. We call such an IFS asymptotically stable.

In this paper we shall show that the set of asymptotically stable IFS's is residual in the family of IFS's, satisfying a Dini condition, acting on a closed, convex subset of a finitely dimensional Banach space. Our proof is based on the result of Barnsley et al. [1]. This paper is an extension of the results of Lasota and Myjak [5] who studied iterated function systems, with constant probabilities, acting on a convex, compact set.

The study of generic properties of nonexpansive transformations, especially Markov operators, has a long history. Main results and a vast literature can be found in [2, 3, 6, 8, 9]. Our result is the first concerning generic properties of Markov operators generated by iterated function systems with state dependent probabilities.

1. Notation and preliminaries. Let (X, ρ) be a metric space such that every closed ball in X

$$B(x,r) = \{y \in X : \rho(x,y) \leqslant r\}$$

is a compact set. We denote by $\mathcal{B}(X)$ the σ–algebra of Borel subsets of X. The space of all finite Borel measures (non-negative, σ–additive) on X will be denoted by $\mathcal{M}$. The subspace of $\mathcal{M}$ which contains only normalized measures (i.e. $\mu(X) = 1$, $\mu \in \mathcal{M}$) will be denoted by $\mathcal{M}_1$ and the elements of this set will be called distributions.

A mapping $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* (see [4,7]) if it satisfies the following two conditions:

$(P\mu)(X) = \mu(X)$ for every $\mu \in \mathcal{M}$;
$P(\alpha_1\mu_1 + \alpha_2\mu_2) = \alpha_1 P\mu_1 + \alpha_2 P\mu_2$ for $\mu_1, \mu_2 \in \mathcal{M}$; $\alpha_1, \alpha_2 \geqslant 0$.

Every Markov operator can be easily extended to the space of signed measures

$$\mathcal{M}_{sig} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}\}.$$

Namely, for every $\nu \in \mathcal{M}_{sig}$, we define

$$P\nu = P\mu_1 - P\mu_2 \quad \text{where} \quad \nu = \mu_1 - \mu_2; \quad \mu_1, \mu_2 \in \mathcal{M}.$$

It is easy to verify that this definition of $P\nu$ does not depend on the choice of $\mu_1, \mu_2 \in \mathcal{M}$.

As usually by $C(X)$ we denote the space of all bounded continuous functions $f : X \to \mathbf{R}$ with the norm

$$\|f\|_c = \sup_{x \in X} |f(x)|.$$

For $\mu \in \mathcal{M}_{sig}$ we define the Fortet-Mourier norm by setting

$$\|\mu\|_{\mathcal{F}} = \sup\{|<f,\mu>| : f \in \mathcal{F}\}, \quad \text{where} \quad <f,\mu> = \int_X f(x)\mu(dx)$$

and

$$\mathcal{F} = \{f \in C(X) : \|f\|_c \leqslant 1 \text{ and } |f(x) - f(y)| \leqslant \rho(x, y) \text{ for } x, y \in X\}.$$

The space $\mathcal{M}_1$ with the distance $\|\mu_1 - \mu_2\|_{\mathcal{F}}$ is a complete metric space and the convergence

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{F}} = 0 \text{ for } \mu_n, \mu \in \mathcal{M}_1$$

is equivalent to the condition

$$\lim_{n\to\infty} < f, \mu_n > = < f, \mu > \quad \text{for all } f \in C(X).$$

Let P be a Markov operator; a measure $\mu \in \mathcal{M}$ is called *stationary* or *invariant* measure of P if $P\mu = \mu$. A Markov operator is called *asymptotically stable* if there exists a stationary distribution $\mu_* \in \mathcal{M}_1$ of P such that

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\|_{\mathcal{F}} = 0 \text{ for } \mu \in \mathcal{M}_1.$$

We say that a Markov operator is *nonexpansive* if

$$\|P\mu_1 - P\mu_2\|_{\mathcal{F}} \leqslant \|\mu_1 - \mu_2\|_{\mathcal{F}} \text{ for } \mu_1, \mu_2 \in \mathcal{M}.$$

Let $(Y, \| \ \|)$ be a Banach space such that $(X, \rho) \subset (Y, \| \ \|)$ and $\rho(x, y) = \|x - y\|$ for $x, y \in X$. Denote by $C(X, Y)$ the set of all continuous transformations acting on X with values in Y. Let $x_0 \in X$ be a fixed point of the space X. For each positive integer k consider a ball $B_k = B(x_0, k)$.

The set $C(X, Y)$, with the metric given by formula

$$(1.1) \quad \rho_*(S, T) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, \sup_{x \in B_k} \|S(x) - T(x)\|\}, \quad \text{for } S, T \in C(X, Y),$$

is a complete metric space. Moreover, the convergence with respect to this metric has the following property: if T_n, T are elements of $C(X, Y), n \in \mathbf{N}$, then

$$(1.2) \quad \lim_{n\to\infty} \rho_*(T_n, T) = 0 \text{ iff } \lim_{n\to\infty} \sup_{x \in B_k} \|T_n(x) - T(x)\| = 0 \text{ for every } k \in \mathbf{N}.$$

For a given $S \in C(X, Y)$ we define

$$(1.3) \qquad L(S) = \sup\{\frac{\|S(x) - S(y)\|}{\|x - y\|} : x, y \in X, x \neq y\}.$$

We call a transformation S Lipschitzian, if the quantity $L(S)$ is finite. We denote by $\mathrm{Lip}(X,Y)$ the set of all Lipschitzian transformations from $C(X,Y)$. Observe that for an arbitrary real number α and $S,T \in \mathrm{Lip}(X,Y)$ we have

$$(1.4) \qquad L(S+T) \leqslant L(S) + L(T) \quad \text{and} \quad L(\alpha S) = |\alpha| L(S).$$

For two Lipschitzian transformations S,T we define

$$(1.5) \quad d_0(S,T) = L(S-T) + \rho_*(S,T), \quad \text{where} \quad \rho_*(S,T) \text{ is given by } (1.1).$$

The set $\mathrm{Lip}(X,Y)$ endowed with the metric d_0 is a complete metric space.

Finally recall that a subset of a metric space $\mathfrak{X}$ is called *residual* if its complement is a set of first Baire category. A property is said to be satisfied by *most elements* of a complete space $\mathfrak{X}$, if it is satisfied on a residual subset. Such a property is also called *generic.*

2. Iterated Function Systems. We will consider some special properties of Markov operators describing the evolutions of measures due to the action of randomly chosen transformations. Fix an integer $N \geqslant 1$.

By an Iterated Function System (shortly IFS) $(S_1, ..., S_N; p_1, ..., p_N)$ we mean a finite sequence of continuous transformations

$$(2.1) \qquad S_i : X \to X \quad \text{for} \quad i = 1, 2, ..., N$$

and a probabilistic vector

$$(2.2) \qquad p_i : X \to [0,1] \quad \text{for} \quad i = 1, ..., N.$$

We will always assume that the functions p_i are continuous and that

$$(2.3) \qquad \sum_{i=1}^{N} p_i(x) = 1, \quad p_i(x) \geqslant 0 \quad \text{for} \quad x \in X, \quad i = 1, ..., N.$$

The iterated function system (2.1), (2.2) will be briefly denoted by (S,p) or sometimes S.

For a given IFS (S,p) we define the *transition operator* $P_S : \mathcal{M} \to \mathcal{M}$ by the formula

$$P_S\mu(A) = \sum_{i=1}^{N} \int_{S_i^{-1}(A)} p_i(x)\mu(dx) \quad \text{for} \quad A \in \mathcal{B}(X) \quad \text{and} \quad \mu \in \mathcal{M}.$$

Evidently, P_S is a Markov operator. It has the property that $\mu_{n+1} = P\mu_n$ where μ_n is the sequence of measures described in the introduction.

We say that the IFS (S,p) is asymptotically stable (or that $\mu_S \in \mathcal{M}_1$ is a stationary measure for (S,p)) if P_S is asymptotically stable (or μ_S is a stationary measure of P_S).

Now we will formulate assumptions that ensure the nonexpansivness and asymptotic stability. We say that a probabilistic vector $(p_1,...,p_N)$ satisfies the *Dini condition* if there is a function $\omega : [0,\infty) \to [0,\infty)$ with the following properties:

(i) ω is continuous and $\omega(0) = 0$;

(ii) ω is nondecreasing and concave, i.e.

$$\alpha\omega(t_1) + (1-\alpha)\omega(t_2) \leqslant \omega(\alpha t_1 + (1-\alpha)t_2) \quad \text{for} \quad t_1, t_2 \geqslant 0, \alpha \in [0,1];$$

(iii) ω is a modulus of continuity for p_i, i.e.

$$\sum_{i=1}^{N} |p_i(x) - p_i(y)| \leqslant \omega(\rho(x,y)) \quad \text{for} \quad x,y \in X \tag{2.4}$$

moreover

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty. \tag{2.5}$$

If, for an IFS (S,p), there is a function ω which satisfies the above conditions, we will call it a *Dini function* of (S,p).

The following lemma was proved in [7] (see also [1]).

LEMMA 1. *Assume that the IFS (S,p) satisfies the inequality*

$$\sum_{i=1}^{N} p_i(x)\rho(S_i(x), S_i(y)) \leqslant r\rho(x,y) \quad \text{for} \quad x,y \in X, \tag{2.6}$$

where $r < 1$ is a non-negative constant.

If there exists a Dini function of (S,p), then there exists a continuous increasing concave function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(\infty) = \infty$ and the Markov operator P_S corresponding to (S,p) is nonexpansive with respect to the metric $\varphi(\rho(x,y))$, i.e.

$$\|P_S\mu_1 - P_S\mu_2\|_{\mathcal{F}_\varphi} \leqslant \|\mu_1 - \mu_2\|_{\mathcal{F}_\varphi} \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}_1,$$

where $\mathcal{F}_\varphi = \{f \in C(X) : \|f\|_c \leqslant 1,\ |f(x) - f(y)| \leqslant \varphi(\rho(x,y)) \text{ for } x,y \in X\}$.

We call the IFS nonexpansive if there is a function φ, described in the above lemma, such that the corresponding Markov operator is nonexpansive with respect to the metric $\varphi \circ \rho$.

We present here a criterion of the asymptotic stability in the version proved in [1], which is the most convenient for our applications.

THEOREM 1. *Let (S,p) be an iterated function system satisfying the following conditions:*

1) *there is a Dini function of (S,p);*
2) $\inf_{x\in X} p_i(x) > 0$ *for every* $i \in \{1,...,N\}$;
3) *the transformations* $S_i : X \to X$ *are Lipschitzian,* $i = 1,...,N$ *and there exists a non-negative constant* λ_S *such that*

$$\sum_{i=1}^{N} p_i(x)L(S_i) \leqslant \lambda_S < 1 \quad \text{for} \quad x \in X.$$

Under the above assumptions the system (S,p) is asymptotically stable.

The conditions in the last theorem was replaced in [7] by some weaker conditions, for instance the second is of the form $p_i(x) > 0$ for every $x \in X$, but we do not use them. In all these investigations the crucial role is played by the number

$$\lambda_S = \sup_{x\in X} \sum_{i=1}^{N} p_i(x)L(S_i). \tag{2.7}$$

3. Generic properties. In this part of the paper we assume that X is a closed convex subset of a Banach space Y. We denote by $\mathfrak{R}$ the set of all IFS satisfying the following conditions:

(1) $S_i \in \mathrm{Lip}(X,X)$, $i = 1,...,N$;
(2) $\sum_{i=1}^{N} p_i(x)L(S_i) \leqslant 1$ for all $x \in X$;
(3) there is a Dini function ω_p of (S,p).

We define a metric in the set $\mathfrak{R}$. Namely, for two given IFS $(S,p),(T,q) \in \mathfrak{R}$, we set

$$d((S,p),(T,q)) = \sum_{i=1}^{N} d_0(S_i,T_i) + \sum_{i=1}^{N} d_1(p_i,q_i), \tag{3.1}$$

where $d_0(S_i,T_i)$ is described by (1.5) and

$$d_1(p_i,q_i) = \sup\{|p_i(x) - q_i(x)| : x \in X\} \quad \text{for } i = 1,...,N.$$

The space $\mathfrak{R}$ can be treated as a subset of the Cartesian product of $\mathrm{Lip}(X,Y)^N$ and $C(X)^N$ endowed with the metric d. In general, the space $\mathfrak{R}$ may not be a complete metric space, so we shall consider special subspaces of the space $\mathfrak{R}$.

Let K be an arbitrary non-negative constant. We will consider the following subset of the space $\mathfrak{R}$

$$\mathfrak{R}_K = \{(S,p) \in \mathfrak{R} : \omega_p(1) \leqslant K \quad \text{and} \quad \int_0^1 \frac{\omega_p(t)}{t} dt \leqslant K\}.$$

We have the following result

THEOREM 2. *The space $(\mathfrak{R}_K, d)$ is a complete metric space.*

PROOF. Consider a sequence of iterated function systems $(S,p)^n = (S_1^n, ..., S_N^n; p_1^n, ..., p_N^n)$ from the space $\mathfrak{R}_K$ and suppose that this sequence satisfies the Cauchy condition with respect to the metric d. Since the space $\mathrm{Lip}(X,Y)^N \times C(X)^N$ is complete in the metric d, this sequence is convergent to an element $(S,p) = (S_1, ..., S_N; p_1, ..., p_N)$. Observe that (S,p) is an IFS and $S_i \in \mathrm{Lip}(X,X)$. For every n we also have

$$\sum_{i=1}^{N} p_i^n(x) L(S_i^n) \leqslant 1 \quad \text{for all} \quad x \in X \tag{3.2}$$

and $\lim_{n\to\infty} L(S_i^n) = L(S_i)$. Thus letting n to infinity in (3.2) we obtain condition (2) for (S,p).

Moreover, for every n there is a Dini function of $(S,p)^n$ denoted by ω_n. From our assumption this function is concave and nondecreasing. Thus

$$\omega_n(t) \leqslant \omega_n(1)t + \omega_n(1). \tag{3.3}$$

We define, for $t \geqslant 0$,

$$\omega(t) := \liminf_{n\to\infty} \omega_n(t). \tag{3.4}$$

According to (3.3) and the assumption $\omega_n(1) \leqslant K$ the function $\omega : [0,\infty) \to [0,\infty)$ is well defined and is also concave and nondecreasing. Of course ω is a modulus of continuity for $(p_1, ..., p_N)$ because every function ω_n satisfies (2.4). To show condition (2.5), observe that the inequality

$$\liminf_{n\to\infty} \int_0^1 \frac{\omega_n(t)}{t} dt \leqslant K$$

and the Fatou lemma imply

$$\int_0^1 \frac{\omega(t)}{t} dt = \int_0^1 \liminf_{n\to\infty} \frac{\omega_n(t)}{t} dt \leqslant \liminf_{n\to\infty} \int_0^1 \frac{\omega_n(t)}{t} dt \leqslant K. \tag{3.5}$$

Therefore, in order to finish the proof it is sufficient to show the continuity of the function ω.

Let $0 < t_0 < 1$ be fixed. For arbitrary $t > t_0$ we have $\omega(t) \geqslant \omega(t_0)$. Since the function ω is non-negative we can write

$$\int_0^1 \frac{\omega(t)}{t} dt \geqslant \int_{t_0}^1 \frac{\omega(t)}{t} dt = \omega(t_0) \ln(\frac{1}{t_0}). \tag{3.6}$$

From (3.6) it follows that the function ω is continiuous at the point $t = 0$. This fact in combination with the concavity of the function ω implies the continuity of this function at every point. Thus (S, p) belongs to $\mathfrak{R}_K$. □

Let $\mathfrak{D}$ be the set of IFS's $(S, p) \in \mathfrak{R}$ having the following properties:

(I) $\lambda_S = \sup_{x \in X} \sum_{i=1}^{N} p_i(x) L(S_i) < 1$;

(II) $p_S = \min_{1 \leqslant i \leqslant N} \inf_{x \in X} p_i(x) > 0$.

Observe that, according to Lemma 1 and Theorem 1, every element from the set $\mathfrak{D}$ is nonexpansive and asymptotically stable.We denote by $\mathfrak{D}_K$ the set of those elements from $\mathfrak{D}$ which belong to $\mathfrak{R}_K$.

LEMMA 2. *The set $\mathfrak{D}_K$ is dense in the space $\mathfrak{R}_K$ endowed with the metric d.*

PROOF. Fix $z \in X$ and $\alpha \in (0, 1)$. Since X is a convex set, for arbitrary transformation $S : X \to X$ the transformation S^α given by the formula

$$S^\alpha(x) = \alpha z + (1 - \alpha) S(x) \quad \text{for} \quad x \in X$$

maps X into itself and

$$L(S^\alpha - S) = \alpha L(S). \tag{3.7}$$

Moreover, using the identity $S^\alpha(x) - S(x) = \alpha(z - S(x))$ for $x \in X$ and the property (1.2) of the metric ρ_* defined by (1.1) we infer that

$$\lim_{\alpha \to 0} \rho_*(S^\alpha, S) = 0,$$

which together with (3.7) gives

$$\lim_{\alpha \to 0} d_0(S^\alpha, S) = 0.$$

We also have $L(S^\alpha) = (1 - \alpha) L(S)$. Therefore the set $\tilde{\mathfrak{D}}_K$ of all iterated function systems (S, p) satisfying (I) is dense in the space $\mathfrak{R}_K$. In order to complete the proof it is sufficient to show that $\mathfrak{D}_K$ is dense in $\tilde{\mathfrak{D}}_K$.

Let $(S, p) \in \tilde{\mathfrak{D}}_K$ and $\varepsilon > 0$ be given. For every $i \in \{1, ..., N\}$ and $x \in X$ we define

$$q_i^\varepsilon(x) = \frac{1}{1 + \varepsilon}\left(p_i(x) + \frac{\varepsilon}{N}\right). \tag{3.8}$$

Of course (S, q^ε) is an IFS fulfilling condition (II). A simple calculation shows that

$$\sup_{x \in X} \sum_{i=1}^{N} q_i^\varepsilon(x) L(S_i) \leqslant \frac{\lambda_S}{1 + \varepsilon} + \frac{\varepsilon}{(1 + \varepsilon) N} \sum_{i=1}^{N} L(S_i). \tag{3.9}$$

Moreover, if ω_p is a Dini function of (S,p) then it is also a Dini function of (S,q^ε) because

$$\sum_{i=1}^{N}|q_i^\varepsilon(x)-q_i^\varepsilon(y)|=\frac{1}{1+\varepsilon}\sum_{i=1}^{N}|p_i(x)-p_i(y)|\leqslant\omega_p(\rho(x,y))\quad\text{for}\quad x,y\in X.$$

Further, since $\sup_{x\in X} p_i(x)\leqslant 1$, we obtain from (3.8)

$$d_1(p_i,q_i)=\sup_{x\in X}\frac{\varepsilon}{1+\varepsilon}\left|\frac{1}{N}-p_i(x)\right|\leqslant\frac{2\varepsilon}{1+\varepsilon} \tag{3.10}$$

for every $i\in\{1,...,N\}$. According to (3.9) the IFS (S,q^ε) belongs to $\tilde{\mathfrak{D}}_K$ for ε sufficiently small, which together with (3.10) completes the proof. □

Observe that the above proof remains the same in the case of the set $\mathfrak{D}$ and the space $\mathfrak{R}$. Thus, the set $\mathfrak{D}$ is dense in $\mathfrak{R}$. Now we state a simple auxiliary lemma.

LEMMA 3. *Let (S,p) and (T,q) be arbitrary elements of the space $\mathfrak{R}$. Then*

$$\lambda_T\leqslant(1+\max_{1\leqslant i\leqslant N}L(S_i))d_0((S,p),(T,q))+\lambda_S. \tag{3.11}$$

PROOF. Fix $x\in X$. For every $i\in\{1,...,N\}$ we have

$$q_i(x)L(T_i)\leqslant q_i(x)L(T_i-S_i)+q_i(x)L(S_i).$$

Since $q_i(x)\leqslant 1$, we may rewrite the last condition in the form

$$q_i(x)L(T_i)\leqslant L(T_i-S_i)+d_1(q_i,p_i)L(S_i)+p_iL(S_i),$$

which leads to the required inequality. □

THEOREM 3. *The set of all $(S,p)\in\mathfrak{R}_K$ which are asymptotically stable and nonexpansive is residual in $\mathfrak{R}_K$.*

PROOF. Define

$$\tilde{\mathfrak{R}}_K=\bigcup_{(S,p)\in\mathfrak{D}_K}B_K((S,p),\delta_S),$$

where $B_K((S,p),\delta_S)$ is an open ball in $\mathfrak{R}_K$ with center at (S,p) and radius

$$\delta_S=p_S\frac{1-\lambda_S}{2(1+L_S)}, \tag{3.12}$$

where $L_S=\max\{L(S_i):1\leqslant i\leqslant N\}$ and λ_S,p_S are described in (I),(II).

Evidently, $\tilde{\mathfrak{R}}_K$ is an open set containing the dense subset $\mathfrak{D}_K$ so it is a residual set. We are going to show that every element from $\tilde{\mathfrak{R}}_K$ is asymptotically stable. To do this let $(T,q) \in \tilde{\mathfrak{R}}_K$. Then there is $(S,p) \in D_K$ such that

$$d((S,p),(T,q)) < \delta_S. \tag{3.13}$$

By virtue of Lemma 3 and (3.12) we obtain

$$\lambda_T \leqslant \frac{1}{2} p_S(1-\lambda_S) + \lambda_S \leqslant \frac{1}{2}(1+\lambda_S) < 1, \tag{3.14}$$

which, according to Lemma 1, implies that IFS (T,q) is nonexpansive. Moreover, for every $i \in \{1,...,N\}$ we have $\sup_{x\in X} |q_i(x) - p_i(x)| < \delta_S$, and this leads to

$$\inf q_i(X) \geqslant p_S - \delta_S = p_S(\frac{1+2L_S+\lambda_S}{2(1+L_S)}) > 0. \tag{3.15}$$

Consequently, in view of (3.14) and (3.15), the assumptions of Theorem 1 are fulfilled. Thus the IFS (T,q) is asymptotically stable, which completes the proof. □

COROLLARY 1. *The set $\mathfrak{R}_a$ of all asymptotically stable iterated function systems $(S,p) \in \mathfrak{R}$ is residual in $\mathfrak{R}$.*

PROOF. Fix $(S,p) \in \mathfrak{D}$ and let the number δ_S be defined by (3.14). Consider the set

$$\tilde{\mathfrak{R}} = \bigcup_{(S,p)\in\mathfrak{D}} B((S,p),\delta_S),$$

where $B((S,p),\delta_S)$ is an open ball in the space $\mathfrak{R}$. Hence the set $\tilde{\mathfrak{R}}$ is residual and by the same arguments, as in the proof above, this set is contained in $\mathfrak{R}_a$. □

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