

## THE SHADOWING LEMMA IN THE LINEAR CASE

BY JERZY OMBACH

1. The Shadowing Lemma says that hyperbolicity implies pseudo-orbits tracing property (shadowing property). A number of proofs of this statement have appeared in various contexts. Roughly, there are two methods of the proof. The analytic one we will discuss here comes from Anosov and Katok, see [4]. It has been repeated or rediscovered in [11], [8], [5] and other papers. Yet, certain technical details which appear in particular situations may cover up the main idea of the proof. The aim of this note is to demonstrate the basic idea of the analytic proof of the Shadowing Lemma in the simplest but nontrivial situation. Namely, we prove that hyperbolic linear homeomorphisms in Banach spaces have the pseudo-orbits tracing property (POTP). We also show that the converse theorem is true in  $\mathbf{R}^n$ . The first statement in  $\mathbf{R}^n$  was proved in [6] in a rather geometric way. The second one was proved in [6] and corrected in [3]. Still, we think our proof is shorter and simpler. At the end of the paper we establish some generalizations of the above results. We want to stress that the core of analytic method is, in fact, Proposition 2 below.

Let  $(X, d)$  be a metric space,  $f: X \rightarrow X$  be a homeomorphism. An orbit of a point  $x \in X$  is a sequence  $\{f^n x\}_{n \in \mathbf{Z}}$ . Let  $\delta > 0$  be fixed. A sequence  $\{x_n\}_{n \in \mathbf{Z}}$  is a  $\delta$ -pseudo-orbit if  $d(fx_n, x_{n+1}) \leq \delta$  for all  $n \in \mathbf{Z}$ . The homeomorphism  $f$  has the POTP if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that given a  $\delta$ -pseudo-orbit  $\{x_n\}_{n \in \mathbf{Z}}$  there is a point  $x \in X$  such that  $d(f^n x, x_n) \leq \varepsilon$  for all  $n \in \mathbf{Z}$ .  $f^n$  means the  $n$ -th iteration of  $f$  i.e.  $f^0 = \text{id}_X$ ,  $f^{n+1} = f^n \circ f$ ,  $n \in \mathbf{Z}$ .

Let  $X$  be a Banach space,  $f: X \rightarrow X$  be a linear homeomorphism. The function  $f$  is hyperbolic if its spectrum is disjoint with the unit circle.

THEOREM 1. *A hyperbolic linear homeomorphism in a Banach space has the POTP.*

THEOREM 2. *A linear homeomorphism in  $\mathbb{R}^n$  having the POTP is hyperbolic.*

2. First we state here three lemmas. The straightforward proofs are omitted.

LEMMA 1. *Let two metrics  $d_1$  and  $d_2$  generate the same uniform structure on the space  $X$ . Then, the homeomorphism  $f$  has the POTP with respect to  $d_1$  iff it has the POTP with respect to  $d_2$ .*

LEMMA 3. *If  $f$  has the POTP and is uniformly continuous, then  $f^{-1}$  has the POTP.*

LEMMA 3. *Let  $(X_i, d_i)$ ,  $i = 1, 2$  be metric space and  $f_i: X_i \rightarrow X_i$  be homeomorphisms. Let  $d$  be a metric on the product space  $X = X_1 \times X_2$ , compatible with the uniform product structure. Let  $f: X \rightarrow X$  be defined by  $F(x_1, x_2) = (f_1 x_1, f_2 x_2)$ . Then both  $f_1$  and  $f_2$  have the POTP iff  $f$  does.*

We recall a well-known result (see for example [1], [2]).

PROPOSITION 1. *Let  $f: X \rightarrow X$  be a linear homeomorphism on a Banach space. Then  $f$  is hyperbolic if and only if the following holds: there exists Banach subspaces  $X_s, X_u \subset X$  and a norm on  $X$  compatible with the original Banach structure such that:  $X = X_s \oplus X_u$ ,  $fX_s = X_s$ ,  $fX_u = X_u$ ,  $\|f|_{X_s}\| < 1$ ,  $\|f^{-1}|_{X_u}\| < 1$ .*

The main step in the proof of Theorem 1 is the following:

PROPOSITION 2. *Let  $X$  be a complete metric space,  $f: X \rightarrow X$  be a contraction (it does not need to be invertible).*

*Then for every  $\varepsilon > 0$  there exists  $\delta > 0$  with the following property. Given a set  $T$ , a bijection  $\tau: T \rightarrow T$  and a map  $\psi: T \rightarrow X$  with*

$$(1) \quad D(f \circ \psi, \psi \circ \tau) \leq \delta$$

*there is the unique map  $\phi: T \rightarrow X$  such that*

$$(2) \quad f \circ \phi = \phi \circ \tau$$

*and*

$$(3) \quad D(\phi, \psi) \leq \varepsilon.$$

Here  $D(\alpha, \beta) = \sup\{d(\alpha(t), \beta(t)): t \in T\}$ .

PROOF. Fix  $\varepsilon > 0$  and define  $\delta = (1 - K)\varepsilon$ , where  $K$  is a contraction constant of  $f$ . Fix a set  $T$ , a bijection  $\tau: T \rightarrow T$  and a map  $\psi: T \rightarrow X$  such that (1) holds. Define

$$X_\varepsilon = \{\alpha: T \rightarrow X: D(\alpha, \psi) \leq \varepsilon\}$$

It is easy to see that  $X_\varepsilon$  is a complete metric space with the distance  $D$  defined above. For any  $\alpha \in X_\varepsilon$  define a map  $F(\alpha): T \rightarrow X$  by

$$F(\alpha) = f \circ \alpha \circ \tau^{-1}.$$

Note that

$$\begin{aligned} D(F(\alpha), \psi) &= D(f \circ \alpha \circ \tau^{-1}, \psi) = D(f \circ \alpha, \psi \circ \tau) \\ &\leq D(f \circ \alpha, f \circ \psi) + D(f \circ \psi, \psi \circ \tau) \leq KD(\alpha, \psi) + \delta \leq \varepsilon. \end{aligned}$$

Thus  $F(\alpha) \in X_\varepsilon$  and so  $F$  is a map from  $X_\varepsilon$  into itself. Moreover,  $F$  is a contraction, as

$$\begin{aligned} D(F(\alpha), F(\beta)) &= D(f \circ \alpha \circ \tau^{-1}, f \circ \beta \circ \tau^{-1}) \\ &= D(f \circ \alpha, f \circ \beta) \leq KD(\alpha, \beta). \end{aligned}$$

The Banach Contraction Principle completes the proof.

**PROPOSITION 3.** *Let  $f: X \rightarrow X$  be a homeomorphism on a complete metric space. If  $f$  is a contraction, then  $f$  has the POTP.*

PROOF. Fix  $\varepsilon > 0$  and pick a  $\delta > 0$  in terms of Proposition 2. Let  $\{x_n\}_{n \in \mathbf{Z}}$  be a  $\delta$ -pseudo-orbit. Define  $T = \mathbf{Z}$ ,  $\tau(n) = n + 1$  and  $\psi(n) = x_n$ . Now, condition (1) in Proposition 2 is satisfied so we get a map  $\phi: \mathbf{Z} \rightarrow Z$  satisfying (2) and (3). Put  $x = \phi(0)$ . Condition (2) implies that  $f^n x = \phi(n)$  for  $z \in \mathbf{Z}$  and by (3) we have  $d(f^n x, x_n) \leq \varepsilon$ , what is desired.

**PROOF OF THEOREM 1.** Apply Proposition 1 to get the decomposition of the Banach space  $X$ . By Proposition 3,  $f|_{X_s}$  and  $f^{-1}|_{X_u}$  have the POTP. By Lemma 2  $f|_{X_u}$  has the POTP and Lemmas 1 and 3  $f$  also has this property.

**PROOF OF THEOREM 2.** Assume that a linear homeomorphism  $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$  has the POTP. Then by Lemma 3 its complexification has the POTP and by the Jordan Decomposition and by Lemmas 3 and 2 maps  $g$  of the form:

$$(4) \quad gx = ax, \quad x \in \mathbf{C}$$

or

$$(5) \quad gx = \begin{bmatrix} a & & & & 0 \\ 1 & a & & & \\ & 1 & & & \\ & & \ddots & & \\ 0 & & & 1 & a \end{bmatrix} x, \quad x \in \mathbf{C}^k,$$

$a$  – an eigenvalue of  $f$ , have this property. We want to show that  $|a| \neq 1$  for any eigenvalue  $a$  of  $f$ . So, assume that  $|a| = 1$  for some eigenvalue  $a$ . If  $g$  is of the form (4) we proceed as Morimoto did in [6]. For a  $\delta > 0$  we define a  $\delta$ -pseudo-orbit  $\{x_n\}$  by  $x_n = na^n\delta$ . It is not traced by any point  $z \in \mathbf{C}$  for

$$|g^n x - x_n| = |a^n x - na^n \delta| = |a^n| |x - n\delta| = |x - n\delta| \rightarrow \infty,$$

as  $n \rightarrow \infty$ . If  $g$  is of the form (5) we proceed as follows.

Define points  $x_n \in \mathbf{C}^k$  by  $x_n = 0$  for  $n \leq 0$  and for  $n \geq 0$  by

$$x_{n+1} = \begin{bmatrix} (n+1)a^{n+1}\delta \\ (gx_n)_2 \\ \vdots \\ (gx_n)_k \end{bmatrix}$$

This sequence is a  $\delta$ -pseudo-orbit as for  $n \geq 0$

$$\|x_{n+1} - gx_n\| = |(n+1)a^{n+1}\delta - na^{n+1}\delta| = |a^{n+1}|\delta = \delta.$$

This  $\delta$ -pseudo-orbit is not traced by any point  $x \in \mathbf{C}^k$  for

$$\begin{aligned} \|g^n x - x_n\| &\geq |(g^n x - x_n)_1| = |a^n(x)_1 - na^n\delta| \\ &= |a^n| |(x)_1 - n\delta| = |(x)_1 - n\delta| \rightarrow \infty, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, the maps of the form (4) and (5) with  $|a| = 1$  do not have the POTP and it is a contradiction.

A continuous linear map (invertible or not) is hyperbolic if its spectrum is disjoint with the unit circle, see for example [9], [10]. If we generalize the concept of the POTP in a quite natural way, one can repeat Theorems 1 and 2 and their proofs with obvious change.

Namely, we establish the following definition of the POTP for maps. Let  $X$  be a metric space and  $f: X \rightarrow X$  a map. A map  $\phi: \mathbf{Z} \rightarrow X$  such that  $\phi(n+1) = f\phi(n)$  for all  $n \in \mathbf{Z}$  is said to be an orbit. A  $\delta$ -pseudo-orbit is a map

$\psi: \mathbf{Z} \rightarrow X$  such that  $d(\psi(n+1), f\psi(n)) \leq \delta$  for all  $n \in \mathbf{Z}$ . The map  $f$  has the POTP if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that given  $\delta$ -pseudo-orbit  $\psi$  there exists an orbit  $\phi$  such that  $D(\phi, \psi) \leq \varepsilon$ .

Another property related to the concept of hyperbolicity is expansiveness. Recall that a homeomorphism  $f: X \rightarrow X$  is expansive if there exists a constant  $e > 0$  such that:  $d(f^n x, f^n y) \leq e$  for all  $n \in \mathbf{Z}$  implies  $x = y$ . It is easy and known that:

- (i) Hyperbolic linear homeomorphisms in Banach spaces are expansive.
- (ii) Expansive linear homeomorphisms in  $\mathbf{R}^n$  are hyperbolic.

Here, we just want to point out that both the above statements and their proofs correspond to Theorems 1 and 2 and their proofs. In Fact, Lemmas 1, 2 and 3 as well as Proposition 3 hold true if we change the POTP for expansiveness.

We show here the proof of the counterpart of Proposition 3 which is essential.

Fix an  $e > 0$  and assume that for some  $x, y \in X$  and all  $n \in \mathbf{Z}$  we have  $d(f^n x, f^n y) \leq e$ . Fix a  $\delta > 0$  to  $\varepsilon = e$  in terms of Proposition 2 and put  $T = \mathbf{Z}$ ,  $\tau(n) = n + 1$ ,  $\phi_1(n) = f^n x$ ,  $\phi_2(n) = f^n y$ . Now,  $\psi = \phi_1$  satisfies condition (1). So, there is the unique map  $\mathbf{Z} \rightarrow X$  satisfying (2) and (3). As both  $\phi_1$  and  $\phi_2$  satisfy these conditions we have  $\phi_1 = \phi_2$ , hence  $x = y$ . It proves the claim and now the proof of statement (i) is the same as the proof of Theorem 1. For the proof of statement (ii) it is enough to consider the complexification and note that for an eigenvector  $x$  with an eigenvalue  $a$ ,  $|a| = 1$  we have  $d(f^n x, f^n 0) = \|a^n x\| = \|x\|$ .

Similarly as in the case of the POTP, expansiveness may be defined for maps, see [7]. Let  $f: X \rightarrow X$  be a map. We say that  $f$  is expansive if there exists a constant  $e > 0$  such that  $D(\phi_1, \phi_2) \leq e$  implies  $\phi_1 = \phi_2$  for orbits  $\phi_1, \phi_2$ . It is easy to see that statements (i) and (ii) are true also for continuous linear maps.

## References

1. Holmes R. B., *Formula for the spectral radius of an operator*, Amer. Math. Monthly **75** (1969), 149–152.
2. Irving M. C., *Smooth Dynamical Systems*, Academic Press, 1980.
3. Kakubari S., *A note on a linear automorphism on  $\mathbf{R}^n$  with the pseudo-orbits tracing property*, Sci. Rep. Nigata Univ. Ser. A no. **23** (1987), 35–37.
4. Katok A., *Local properties of hyperbolic sets (in Russian), appendix to the russian edition of Z. Nitecki, Differentiable dynamics*, Mir, 1975.
5. Meyer K., Sell G., *An analytic proof of the Shadowing Lemma*, Funkcial. Ekvac. **30** (1987), 127–133.
6. Morimoto A., *Manifolds and Lie Groups*, Progress in Math. 14, Birkhäuser, 1981, pp. 283–299.
7. Mrozek M., private communicate.

8. Nadzieja T., *Shadowing Lemma for family of  $\varepsilon$ -trajectories*, Bull. Acad. Polon. Sci., Ser. Sci. math., astr., phys. **26** (1978), 335–341.
9. Ombach J., *On the Stable Manifolds Theorem*, Bull. Acad. Polon. Sci., Ser. Sci. math., astr., phys. **26** (1978), 335–341.
10. Robbin J., *Stable manifolds for semi-hyperbolic fixed points*, Illinois J. Math. **15** (1971), 595–609.
11. Shub M., *Stabilité-globale des systems dynamiques*, Asterisque **56** (1978).

*Received September 25, 1989*

Institute of Mathematics  
Jagiellonian University  
ul. Reymonta 4  
30-059 Kraków, Poland